Theoretical Computer Science 516 (2014) 96–100

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Fast recognition of doubled graphs

Frédéric Maffray

CNRS, Laboratoire G-SCOP, INP, UJF, Grenoble, France

ARTICLE INFO

Article history:
Received 4 March 2013
Accepted 15 November 2013
Communicated by P. Widmayer

Keywords:
Double split graph
Recognition

ABSTRACT

Double split graphs form one of the five basic classes in the proof of the strong perfect graph theorem by Chudnovsky et al. [3]. A doubled graph is any induced subgraph of a double split graph. We show here that deciding if a graph G is a doubled graph can be done in time $O(|V(G)| + |E(G)|)$ by analyzing the degree structure of the graph.

1. Introduction

The main result in the proof of the strong perfect graph theorem by Chudnovsky et al. [3] is a decomposition theorem for all Berge graphs. This theorem states that every Berge graph either admits one of several decompositions or belongs to one of five basic classes. The five basic classes are bipartite graphs, complements of bipartite graphs, line-graphs of bipartite graphs, complements of line-graphs of bipartite graphs, and double-split graphs. There are well-known characterizations by forbidden subgraphs and linear-time recognition algorithms for the first four classes [7–9]. We will focus here on the class of double-split graphs. Before giving the precise definition of this class, we need some terminology and notation.

All graphs considered here are finite and have no loops or multiple edges. Given a graph G, its vertex-set is denoted by $V(G)$ and its edge-set by $E(G)$. The complement of G is denoted by $\overline{G}$. The neighborhood of a vertex v (the set of vertices adjacent to v) is denoted by $N(x)$. The degree of v (the size of its neighborhood) is denoted by $d(v)$. We let P_n and C_n denote the path and cycle on n vertices, respectively. For any $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G induced by S. A *clique* is a set of pairwise adjacent vertices. A *stable set* is a set of pairwise non-adjacent vertices. A vertex v is *complete* to a set $S \subset V(G)$ if v is adjacent to every vertex in S, and *anticomplete* to S if it has no neighbor in S. Given a set $\mathcal{F}$ of graphs, a graph G is $\mathcal{F}$*-free* if no induced subgraph of G is isomorphic to any member of $\mathcal{F}$.

Following [1], we say that a graph G is *semi-matched* if every vertex in G has at most one neighbor (in other words, every component of G has size at most 2). A graph G is *semi-antimatched* if $\overline{G}$ is semi-matched. In a graph G we say that a set $S \subseteq V(G)$ is semi-matched (resp. semi-antimatched) if $G[S]$ is semi-matched (resp. semi-antimatched). Let $X, Y \subseteq V(G)$ be such that $X \cap Y = \emptyset$ and $X \cup Y = V(G)$. We say that the partition (X, Y) is *acceptable* if X is semi-matched, Y is semi-antimatched, and:

(a1) For all adjacent $u, v \in X$, every $y \in Y$ is adjacent to exactly one of u, v;
(a2) For all non-adjacent $v, w \in Y$, every $x \in X$ is adjacent to exactly one of v, w.

A graph is *split* if its vertex-set can be partitioned into a stable set and a clique. A graph is *double-split* if it admits an acceptable partition (X, Y) such that all components of $G[X]$ and of $\overline{G}[Y]$ have size 2 and both $G[X]$ and $\overline{G}[Y]$ have at least two components. A *doubled* graph is any induced subgraph of a double-split graph; in other words, any graph that admits an acceptable partition (X, Y). Note that a graph G is a doubled graph with acceptable partition (X, Y) if and only

E-mail address: frederic.maffray@inpg.fr.

0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2013.11.020

if $\overline{G}$ is a doubled graph with acceptable partition (Y, X). The class of double-split graphs is not closed under taking induced subgraphs, but the class of doubled graphs is, so it can be characterized by its family $\mathcal{F}$ of minimal forbidden induced subgraphs. Alexeev et al. [1] determined the family $\mathcal{F}$; it consists of 44 graphs of order at most 9. We will not describe it here.

Theorem 1.1. *(See [1].) A graph is a doubled graph if and only if it is $\mathcal{F}$-free.*

It follows from Theorem 1.1 that one can decide if a graph G on n vertices is a doubled graph by testing every induced subgraph of G on at most 9 vertices; this takes time $O(n^9)$. Chudnovsky et al. [2, Theorem 5.1] showed that this can be done in time $O(n^5)$ using the definition of a doubled graph. The question of the existence of a linear-time recognition algorithm is raised in [10]. We show that one can decide if a graph G is a doubled graph in time $O(|V(G)| + |E(G)|)$ (linear time) by analyzing the degree sequence of G.

We first observe that, given a partition (X, Y), we can determine if it is acceptable in linear time, as follows. Determine if X is semi-matched by building the graph $G[X]$ and checking that every vertex has degree at most 1 in $G[X]$. Likewise, determine if Y is semi-antimatched by building the graph $G[Y]$ and checking that every vertex has degree at least $|Y| - 2$ in $G[Y]$. Now assume that X is semi-matched and Y is semi-antimatched. Determine if (X, Y) satisfies axiom (a1) in the definition of an acceptable partition as follows. If X is a stable set, there is nothing to do. If X is not a stable set, let $X_1, \ldots, X_h$ be the components of size 2 of $G[X]$. Initialize a counter $(c_1, \ldots, c_h) = (0, \ldots, 0)$. Pick a vertex $y \in Y$ and scan its adjacency list. For every neighbor z of y, if $z \in X_i$ for some $i \in \{1, \ldots, h\}$ then set $c_i := c_i + 1$. If $c_i = 2$, declare that the partition is not acceptable (indeed y has two neighbors in X_i) and stop. When the whole adjacency list of y is scanned and the algorithm has not stopped, check whether there is any j with $c_j = 0$; in that case, declare that the partition is not acceptable (indeed y has no neighbor in X_j) and stop. Else, reinitialize the counter. This procedure takes time $O(d(y))$ (in particular when we reinitialize the counter we have $h \leqslant d(y)$ since y has a neighbor in each X_i). Continue with every vertex in Y. In total this takes time $O(\sum_{y \in Y} d(y))$, which is linear time. Axiom (a2) can be checked similarly.

By the preceding argument, in order to decide if a graph is a doubled graph it suffices to guess a partition (X, Y) and to check whether it is acceptable or not. However, we might have to check $O(2^{|V(G)|})$ partitions. Our main result, Theorem 2.2 below, shows that there is only a constant number of partitions to check, whatever the input graph is. This will ensure that we can decide if a graph is a doubled graph in linear time.

2. The degree sequence

Let G be a graph on n vertices $v_1, \ldots, v_n$ such that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, where $d_i = d(v_i)$ for all i. Let t be the largest integer such that $d_t \leqslant n - t$ (note that t exists because $d_1 \leqslant n - 1$). Define sets:

$$T = \{v_1, \ldots, v_t\},$$
$$D = \{v \mid d(v) \leqslant d_t\},$$
$$D' = \{v \mid d(v) < d_t\},$$
$$D'' = \{v \mid d(v) \leqslant d_t + 1\}.$$

Recall that the degree sequence of a graph can be obtained in linear time, for example with bucket sort [4], so integers t and d_t and sets T, D, D' and D'' can be computed in linear time. Hammer and Simeone [6] observed that split graphs can be recognized in linear time on the basis of the following result.

Theorem 2.1. *(See [6].) Let G be a graph on n vertices $v_1, \ldots, v_n$ such that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, where $d_i = d(v_i)$ for all i. Let t and T be defined as above. Then G is a split graph if and only if T is a stable set and $V(G) \setminus T$ is a clique.*

Our main theorem is a generalization of Theorem 2.1. For this purpose, we need to introduce some more definitions. In a graph G, a 2-*vertex* is a vertex of degree 2 whose neighbors are not adjacent. A 3-*vertex* is a vertex of degree 3 whose neighborhood contains exactly one edge. Let integers t and d_t be defined as above. A d_t-*edge* is an edge whose vertices have degree 1 and d_t respectively. A d_t-*pair* is a pair of non-adjacent vertices of degree $n - 2$ and d_t respectively. As we know the degree sequence, the 2-vertices, 3-vertices, d_t-edges and d_t-pairs of a graph can be found in linear time.

Let us say that a set $X \subseteq V(G)$ is *acceptable* if the partition $(X, V(G) \setminus X)$ is acceptable.

Theorem 2.2. *Let G be a graph on n vertices $v_1, \ldots, v_n$ such that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, where $d_i = d(v_i)$ for all i. Let integers t and d_t and sets T, D, D' and D'' be defined as above. Then G is a doubled graph if and only if any of the candidate sets defined below is acceptable*:

1. *Sets $\emptyset$, $V(G)$, T, D, D' and D'' are candidates.*
2. *If $|D| \leqslant 4$, then every set $X \subseteq D$ with $|X| = 2$ is a candidate.*