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Note

Edge coloring of planar graphs which any two short cycles are adjacent at most once

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ABSTRACT

By applying discharging methods and properties of critical graphs, we proved that every simple planar graph G is of class 1 if $\Delta(G) = 6$ and any k-cycle is adjacent to at most one k-cycle for some k $(k = 3, 4, 5)$.

1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [1] for the terminologies and notations not defined here. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. If a graph G can be drawn in the plane so that each pair of edges intersects only at their ends, it is said to be a *planar graph*. For a plane graph G, we denote its face set by $F(G)$, and for a face $f \in F(G)$, the degree $d(f)$ of a face f is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+-vertex (or face) is a vertex (or face) of degree k, at least k. A k(or k^+)-vertex adjacent to a vertex x is called a k(or k^+)-neighbor of x. A k-cycle is a cycle of length k. Two cycles sharing a common edge are said to be adjacent. Given a cycle C of length k in G, an edge $xy \in E(G)\backslash E(C)$ is called a *chord* of C if $x, y \in V(C)$. Such a cycle C is also called a chordal-k-cycle.

An edge k-coloring of a graph G is a function $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent edges $e_1, e_2 \in E(G)$ have $\phi(e_1) \neq \phi(e_2)$. The chromatic index $\chi'(G)$ is the smallest integer k such that G admits an edge k-coloring. A graph G is of class 1 if $\chi'(G) = \Delta$ and of class 2 if $\chi'(G) = \Delta + 1$, A critical graph G is a connected graph such that G is of class 2 and $\chi'(G - e) < \chi'(G)$ for each edge $e \in E(G)$. A critical graph of maximum degree Δ is called a Δ-critical graph. It is obvious that every Δ-critical graph ($\Delta \geqslant 2$) is 2-connected.

Vizing [2] first presented examples of planar graph of class 2 for each $\Delta \in \{2, 3, 4, 5\}$ showed that every planar graph with $\Delta \geqslant 8$ is of class 1 and conjectured that the conclusion holds for $6 \leqslant \Delta \leqslant 7$. The case $\Delta = 7$ was confirmed by Sanders and Zhao [3], and Zhang [4] independently. Thus, Vizing's conjecture remains open only for the case $\Delta = 6$. The girth of a

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graph G is defined to be the length of a shortest cycle in G. Li and Luo [5] proved that a planar graph G with maximum degree Δ and girth g is of class 1 if it satisfies one of the following conditions: (1) $\Delta \geqslant 3$ and $g \geqslant 8$ (2) $\Delta \geqslant 4$ and $g \geqslant 5$ (3) $\Delta \geqslant 5$ and $g \geqslant 4$ (4) $\Delta \geqslant 8$ and $g \geqslant 3$. Zhou [6] proved that every planar graph with $\Delta = 6$ having no k-cycle for some $k \in \{3, 4, 5\}$ is of class 1. Bu and Wang [7] proved that if G does not either a 6-cycle, or a 4-cycle with a chord, or a 5- and 6-cycle with a chord, then G is of class 1. The result was further extended by Wang and Chen [8] to a planar graph G with $\Delta = 6$ without chordal 5-cycle. Recently, Ni [9,10] proved every simple planar graph with $\Delta = 6$ is of class 1, if it satisfies one of the following conditions: (1) any two k-cycles are not adjacent for some k $(k = 3, 4, 5)$, (2) a planar graph does not contain chordal-7-cycles. For planar graph G with $\Delta = 5$, Chen and Wang [11] proved that the planar graph G without intersecting triangles is of class 1. Wu [12] proved that if G does not contain a 4-cycle, or a 5-cycle, then G is of class 1. Ni [13,14] proved the planar graph G is of class 1, if any 3-cycle is not adjacent to a k-cycle for some $k \in \{4, 5\}$, or any 4-cycle is not adjacent to a 5-cycle. In this paper, we proved that every simple planar graph with $\Delta = 6$ is of class 1, if any k-cycle is adjacent to at most one k-cycle for some k $(k = 3, 4, 5)$.

2. Main result and its proof

First, we give some Lemmas.

Lemma 1. *(See [2].) Let G be a Δ-critical graph and $\Delta \geqslant 3$. Then*

(1) *any vertex of G is adjacent to at most one 2-vertex, and at least two Δ-vertices,*
(2) *if $uv \in E(G)$ with $d(u) = k$, where $k \neq \Delta$, then $d(u) + d(v) \geqslant \Delta + 2$, and v is adjacent to at least $\Delta - k + 1$ Δ-vertices.*

Lemma 2. *(See [1].) Let G be a Δ-critical graph. If $xy \in E(G)$ and $d(x) + d(y) = \Delta + 2$, then*

(1) *any $v \in N(\{x, y\}) \backslash \{x, y\}$ is Δ-vertex,*
(2) *any $v \in N(N(\{x, y\})) \backslash \{x, y\}$ satisfies $d(v) \geqslant \Delta - 1$, and*
(3) *if $d(x) < \Delta$, $d(y) < \Delta$, then any $v \in N(N(\{x, y\})) \backslash \{x, y\}$ is Δ-vertex.*

Lemma 3. *(See [3].) If a graph G has distinct vertices x, y, z such that (1) $xy \in E(G)$, $xz \in E(G)$, $d(z) < 2\Delta - d(x) - d(y) + 2$, and (2) xz is in at least $d(x) + d(y) - \Delta - 2$ triangles not containing y, then G is not a critical graph.*

Then, we began to prove the main result of the paper.

Theorem 1. *If G is a planar graph with $\Delta = 6$ such that any s-cycle is adjacent to at most one s-cycle for some $s (s = 3, 4, 5)$, then G is of class 1.*

Proof. Suppose on the contrary that G is of Class 2. Without loss of generality, we may assume that G is 6-critical. Since G is a planar graph, by Euler's formula, we have

$$\sum_{x \in V(G)} \bigl(d(x) - 4\bigr) + \sum_{x \in F(G)} \bigl(d(x) - 4\bigr) = -8 < 0.$$

We define ch to be the initial charge. Let $ch(x) = d(x) - 4$ for each $x \in V \cup F$. So $\sum_{x \in V \cup F} ch(x) < 0$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V \cup F$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum. If we can show that $ch'(x) \geqslant 0$ for each $x \in V \cup F$, then we get an obvious contradiction $0 \leqslant \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) < 0$. which completes our proof.

Let $d_r(v)$ denote the number of r-neighbors of $v (r = 2, 3, 4, 5, 6)$, and $f_3(v)$ the number of 3-faces incident with v.

Case 1. $s = 3$, that is, every 3-cycle is adjacent to at most one 3-cycle.

The discharging rules are defined as follows.

R11 Let v be a 6-vertex.

R11-1 v sends 1 to its adjacent 2-vertex, sends $\frac{1}{3}$ to each of its adjacent 3-vertices, sends $\frac{2}{9}$ to each of its adjacent 4-vertices, sends $\frac{1}{9}$ to each of its adjacent 5-vertices.

R11-2 If v is adjacent to a 5-vertex y and y is incident with a $(3, 5, 6)$-face f, but v is not incident with f, then v sends $\frac{5}{54}$ by y to the 6-vertex incident with f.

R11-3 If v is adjacent to a 6-vertex u and u is incident with a $(2, 6, 6)$-face f, but v is not incident with f, then v sends $\frac{1}{8}$ to u.

R11-4 If v is incident with a $(3, 6, 6)$-face $[u, v, w]$ such that $d(u) = 3$ and uw is incident with two $(3, 6, 6)$-faces, then v sends $\frac{1}{18}$ to w.