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Complexity of the bisection method

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Abstract

The *bisection method* is the consecutive bisection of a triangle by the median of the longest side. In this paper we prove a subexponential asymptotic upper bound for the number of similarity classes of triangles generated on a mesh obtained by iterative bisection, which previously was known only to be finite. The relevant parameter is γ/σ, where γ is the biggest and σ is the smallest angle of the triangle. We get this result by introducing a taxonomy of triangles that precisely captures the behaviour of the bisection method. We also prove that the number of directions on the plane given by the sides of the triangles generated is finite. Additionally, we give purely geometrical and intuitive proofs of classical results for the bisection method.

Keywords: Bisection method; Longest edge bisection; Triangulation

1. Introduction

Adaptive finite element methods and multigrid algorithms are widely used numerical techniques for the accurate analysis of physical and engineering problems modelled by partial differential equations, both of which require sequences of flexible and quality discretizations of the associated geometric region. Longest edge refinement algorithms for triangulations, based on the (longest edge) bisection of triangles [6,5] were especially designed to deal with these issues: they are able to perform iterative local refinement of the involved triangulations by essentially maintaining the geometrical quality of the input mesh as needed in finite element applications; they produce hierarchies of quality and nested irregular triangulations as required for non-structured multigrid methods. Furthermore because of their simplicity, these algorithms have been also successfully used for the parallel refinement of big meshes in parallel finite element applications [3,6].

Longest edge algorithms work as follows: they perform selective and iterative longest edge bisection of some target triangles and some neighbours in order to produce a conforming mesh (where the intersection of pairs of neighbour triangles is either a common vertex or a common edge). For an illustration of the use of these algorithms, see Fig. 1, where a locally refined triangulation of an input triangulation is shown. The properties of longest-edge algorithms are inherited from the non-degeneracy properties of the iterative longest edge bisection of triangles, which essentially

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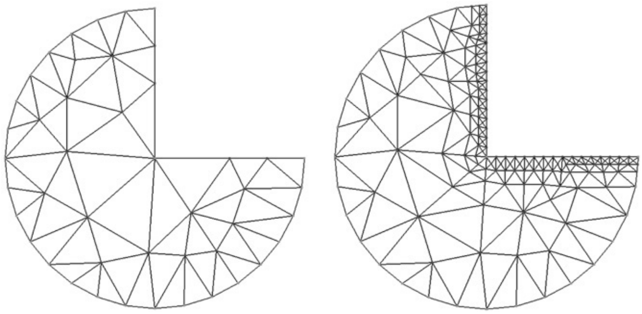

Fig. 1. An initial triangulation (on the left), and a local refinement of it.

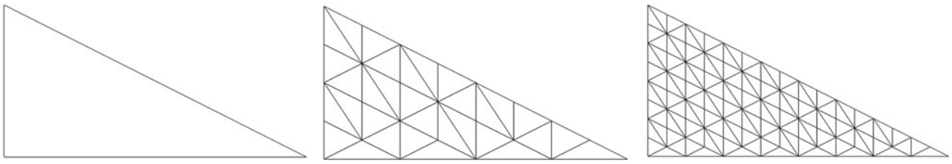

Fig. 2. Refinement by iterated bisections.

guarantee that consecutive bisections of the triangles nested in any triangle t of smallest angle σ produce triangles t' (of minimum angle σ') such that $\sigma' \geq \sigma/2$, and that the number of non-similar triangles generated is finite. See the example in Fig. 2. These properties not only assure that the smallest angles of the refined triangles are bounded, but also guarantee that the refined area is increasingly covered by better triangles as the refinement proceeds. Note, however, that this paper does not study refinement algorithms but the basic properties of the longest edge bisection of an individual triangle (bisection method) throughout the bisection levels.

The systematic study of the bisection method began in a series of papers [2,7–9,1] around two decades ago. First, Rosenberg and Stenger [7] proved that the method does not degenerate the smallest angle of the triangles generated by showing that it does not decrease beyond $\sigma/2$, where σ is the smallest angle from the initial triangle. Then Kearfott [2] proved a bound on the behaviour of the *diameter* (the length of the longest side of any triangle obtained). Later Stynes [8] presented a better bound for certain triangles. This bound was improved independently by Stynes [9] and Adler [1] for all triangles. From their proofs they also deduced that the number of classes of similarity of triangles generated is finite, although they give no bound.

There is very little research so far on complexity aspects of the bisection method. Although it is known that different types of triangles behave radically differently under iterative bisection ("good" and "bad" triangles), no systematic classification of them is known.

This paper attempts to fill these gaps in the analysis of the bisection method. We present a precise taxonomy of triangles that captures the behaviour of the bisection method for different types of triangles. We introduce as our main parameter the smallest angle, and prove that in the plane it predicts faithfully the behaviour of the bisection method. We use this framework to prove new results and to give intuitive proofs of classical results.

The contributions of this paper are as follows:

- A taxonomy of triangles reflecting the behaviour of the bisection method. We consider six classes of triangles, and two main groups.
- An asymptotic bound on the number of non-similar triangles generated. We prove a subexponential asymptotic upper bound, identify the instances where this bound is polynomial, and describe worst-case instances.
- An analysis of lower bounds on the smallest angle of triangles in the mesh obtained using the bisection method for each class of triangles defined.
- A proof that there is a finite number of directions in the plane generated by the corresponding segments (sides) of the triangles generated, and asymptotic bounds on this number.

Additionally, we present a unified view of the main known results for the bisection method from an elementary geometry point of view. This approach allows intuitive proofs and has the advantage of presenting the geometry inherent in the method.