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On the dualization of hypergraphs with bounded edge-intersections and other related classes of hypergraphs[☆]

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Abstract

Given a finite set V, and integers $k \geq 1$ and $r \geq 0$, let us denote by $\mathbb{A}(k, r)$ the class of hypergraphs $\mathcal{A} \subseteq 2^V$ with (k, r)-bounded intersections, i.e. in which the intersection of any k distinct hyperedges has size at most r. We consider the problem MIS$(\mathcal{A}, \mathcal{I})$: given a hypergraph $\mathcal{A}$, and a subfamily $\mathcal{I} \subseteq \mathcal{I}(\mathcal{A})$ of its maximal independent sets (MIS) $\mathcal{I}(\mathcal{A})$, either extend this subfamily by constructing a new MIS $I \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{I}$ or prove that there are no more MIS, that is $\mathcal{I} = \mathcal{I}(\mathcal{A})$. It is known that, for hypergraphs of bounded dimension $\mathbb{A}(1, \delta)$, as well as for hypergraphs of bounded degree $\mathbb{A}(\delta, 0)$ (where δ is a constant), problem MIS$(\mathcal{A}, \mathcal{I})$ can be solved in incremental polynomial time. In this paper, we extend this result to any integers k, r such that $k + r = \delta$ is a constant. More precisely, we show that for hypergraphs $\mathcal{A} \in \mathbb{A}(k, r)$ with $k + r \leq$ const, problem MIS$(\mathcal{A}, \mathcal{I})$ is NC-reducible to the problem MIS$(\mathcal{A}', \emptyset)$ of generating a single MIS for a partial subhypergraph $\mathcal{A}'$ of $\mathcal{A}$. In particular, this implies that MIS$(\mathcal{A}, \mathcal{I})$ is polynomial, and we get an incremental polynomial algorithm for generating all MIS. Furthermore, combining this result with the currently known algorithms for finding a single maximally independent set of a hypergraph, we obtain efficient parallel algorithms for incrementally generating all MIS for hypergraphs in the classes $\mathbb{A}(1, \delta)$, $\mathbb{A}(\delta, 0)$, and $\mathbb{A}(2, 1)$, where δ is a constant. We also show that, for $\mathcal{A} \in \mathbb{A}(k, r)$, where $k + r \leq$ const, the problem of generating all MIS of $\mathcal{A}$ can be solved in incremental polynomial-time and with space polynomial only in the size of $\mathcal{A}$.

Keywords: Bounded degree; Bounded dimension; Conformal hypergraph; Dualization; Incremental generating; Maximal independent set; Minimal transversal; Polynomial space

1. Introduction

Let $\mathcal{A} \subseteq 2^V$ be a hypergraph (set family) on a finite vertex set V. A vertex set $I \subseteq V$ is called *independent* if I contains no hyperedge of $\mathcal{A}$. Let $\mathcal{I}(\mathcal{A}) \subseteq 2^V$ denote the family of all maximal independent sets (MIS) of $\mathcal{A}$.

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[✠] Our friend and co-author, Leonid Khachiyan passed away with tragic suddenness, while we were working on the final version of this paper.

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We assume that $\mathcal{A}$ is given by the list of its hyperedges, and consider the problem *GEN-MIS*($\mathcal{A}$) of incrementally generating all sets in $\mathcal{I}(\mathcal{A})$:

GEN-MIS($\mathcal{A}$): *Given a hypergraph $\mathcal{A}$, generate all maximal independent sets of $\mathcal{A}$.*

Clearly, this problem can be solved by performing $|\mathcal{I}(\mathcal{A})| + 1$ calls to the following problem:

MIS($\mathcal{A}, \mathcal{I}$): *Given a hypergraph $\mathcal{A}$ and a collection $\mathcal{I} \subseteq \mathcal{I}(\mathcal{A})$ of its maximal independent sets, either find a new maximal independent set $I \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{I}$, or prove that the given collection is complete, $\mathcal{I} = \mathcal{I}(\mathcal{A})$.*

Note that if $I \in \mathcal{I}(\mathcal{A})$ is an independent set, the complement $B = V \setminus I$ is a *transversal* to $\mathcal{A}$; that is, $B \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, and vice versa. Hence $\{B \mid B = V \setminus I,\ I \in \mathcal{I}(\mathcal{A})\} = \mathcal{A}^d$, where

$$\mathcal{A}^d \stackrel{\text{def}}{=} \{B \mid B \text{ is a minimal transversal to } \mathcal{A}\}$$

is the *transversal* or *dual* hypergraph of $\mathcal{A}$. For this reason, the problems GEN-MIS($\mathcal{A}$) and MIS($\mathcal{A}$) can be equivalently stated as the following *hypergraph dualization problems*:

GEN-DUAL($\mathcal{A}$): *Given a hypergraph $\mathcal{A}$, generate all minimal transversals of $\mathcal{A}$.*

DUAL($\mathcal{A}, \mathcal{B}$): *Given a hypergraph $\mathcal{A}$ and a collection $\mathcal{B} \subseteq \mathcal{A}^d$ of minimal transversals to $\mathcal{A}$, either find a new minimal transversal $B \in \mathcal{A} \setminus \mathcal{B}$ or show that $\mathcal{B} = \mathcal{A}$.*

These problems have applications in combinatorics, graph theory, artificial intelligence, game theory [18,19,26], reliability theory, database theory, integer programming, and learning theory (see, e.g. [5,11]). It is an open question as to whether the problem DUAL($\mathcal{A}, \mathcal{B}$), or equivalently MIS($\mathcal{A}, \mathcal{I}$), can be solved in polynomial time for arbitrary hypergraphs. The fastest currently known algorithm [14] for DUAL($\mathcal{A}, \mathcal{B}$) is quasi-polynomial and runs in time $O(nm) + m^{o(\log m)}$, where $n = |V|$ and $m = |\mathcal{A}| + |\mathcal{B}|$. The fastest known randomized parallel algorithm [22], for problem MIS($\mathcal{A}, \emptyset$) of computing a single MIS of a hypergraph $\mathcal{A}$ on n vertices, runs in time $O(\sqrt{n})$ on $n^{3/2}$ processors.

It was shown in [6,11] that in the case of hypergraphs of bounded dimension,

$$\dim(\mathcal{A}) \stackrel{\text{def}}{=} \max_{A \in \mathcal{A}} |A| \leq \text{const.} \tag{1}$$

problem MIS($\mathcal{A}, \mathcal{I}$) can be solved in polynomial time. Moreover, [4] shows that the problem can be efficiently solved in parallel, MIS($\mathcal{A}, \mathcal{I}$) $\in NC$ for $\dim(\mathcal{A}) \leq 3$ and MIS($\mathcal{A}, \mathcal{I}$) $\in RNC$ for $\dim(\mathcal{A}) = 4, 5, \ldots$. Let us also mention that for graphs, $\dim(\mathcal{A}) \leq 2$, all MIS can be generated with polynomial delay, see [20] and also [29].

In [10], a total polynomial time generation algorithm was obtained for hypergraphs of bounded degree,

$$\deg(\mathcal{A}) \stackrel{\text{def}}{=} \max_{v \in V} |\{A \ :\ v \in A \in \mathcal{A}\}| \leq \text{const.} \tag{2}$$

This result was recently strengthened in [12], where a polynomial delay algorithm was obtained for a wider class of hypergraphs.

In this paper, we consider the class $\mathbb{A}(k, r)$ of hypergraphs with (k, r)-*bounded intersections:* $\mathcal{A} \in \mathbb{A}(k, r)$ if the intersection of each (at least) k distinct hyperedges of $\mathcal{A}$ is of cardinality at most r. We will always assume that $k \geq 1$ and $r \geq 0$ are fixed integers whose sum is bounded, $k + r \leq \delta = \text{const}$. Note that

$$\dim(\mathcal{A}) \leq r \quad \text{iff} \quad \mathcal{A} \in \mathbb{A}(1, r) \quad \text{and} \quad \deg(\mathcal{A}) < k \quad \text{iff} \quad \mathcal{A} \in \mathbb{A}(k, 0),$$

and hence the class $\mathbb{A}(k, r)$ contains both the bounded-dimension and bounded-degree hypergraphs as subclasses.

It will be shown that problem MIS($\mathcal{A}, \mathcal{I}$) can be solved in polynomial time for hypergraphs with (k, r)-bounded intersections. It is not difficult to see that for any hypergraph $\mathcal{A} \in \mathbb{A}(k, r)$, the following property holds for every vertex-set $X \subseteq V$ (see Lemma 1 below): X is contained in a hyperedge of $\mathcal{A}$ whenever each subset of X of cardinality at most $\delta = k + r$ is contained in a hyperedge of $\mathcal{A}$. Hypergraphs $\mathcal{A} \subseteq 2^V$ with this property were introduced by Berge [3] under the name of δ-*conformal hypergraphs*, and clearly define a wider class of hypergraphs than $\mathbb{A}(k, r)$ with $k + r = \delta$. In fact, we will prove our results for this wider class of δ-*conformal* hypergraphs.

Theorem 1. *For the δ-conformal hypergraphs, $\delta \leq$ const, and in particular for $\mathcal{A} \in \mathbb{A}(k, r)$, $k + r \leq \delta =$ const, problem MIS($\mathcal{A}, \mathcal{I}$) can be solved in polynomial time. Hence $\mathcal{I}(\mathcal{A})$, the set all MIS of $\mathcal{A}$, can be generated in incremental polynomial time.*