Theoretical Computer Science 506 (2013) 115–121

Contents lists available at SciVerse ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Solving MIN ONES 2-SAT as fast as VERTEX COVER ☆

Neeldhara Misra [a], N.S. Narayanaswamy [c,1], Venkatesh Raman [b,*], Bal Sri Shankar [b]

[a] Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India
[b] Institute of Mathematical Sciences, Chennai, India
[c] Indian Institute of Technology, Madras, India

ARTICLE INFO

Article history:
Received 12 March 2012
Received in revised form 26 April 2013
Accepted 18 July 2013
Communicated by J. Kratochvil

Keywords:
MIN ONES 2-SAT
VERTEX COVER
Linear programming relaxation
Parameterized complexity

ABSTRACT

The problem of finding a satisfying assignment that minimizes the number of variables that are set to 1 is NP-complete even for a satisfiable 2-SAT formula. We call this problem MIN ONES 2-SAT. It generalizes the well-studied problem of finding the smallest vertex cover of a graph, which can be modeled using a 2-SAT formula with no negative literals. The natural parameterized version of the problem asks for a satisfying assignment of weight at most k. In this paper, we present a polynomial-time reduction from MIN ONES 2-SAT to VERTEX COVER without increasing the parameter and ensuring that the number of vertices in the reduced instance is *equal* to the number of variables of the input formula. Consequently, we conclude that this problem also has a simple 2-approximation algorithm and a $2k - c\log k$-variable kernel subsuming (or, in the case of kernels, improving) the results known earlier. Further, the problem admits algorithms for the parameterized and optimization versions whose runtimes will always match the runtimes of the best-known algorithms for the corresponding versions of vertex cover.

Finally we show that the optimum value of the LP relaxation of the MIN ONES 2-SAT and that of the corresponding VERTEX COVER are the same. This implies that the (recent) results of VERTEX COVER version parameterized above the optimum value of the LP relaxation of VERTEX COVER carry over to the MIN ONES 2-SAT version parameterized above the optimum of the LP relaxation of MIN ONES 2-SAT.

1. Introduction and motivation

Satisfiability is a fundamental problem that encodes several computational problems. Variations of the problem appear as canonical complete problems for several complexity classes. While it is well known that the satisfiability of a formula in CNF form is a canonical NP-complete problem, testing whether a CNF formula has a satisfying assignment with weight[2] k is a canonical complete problem for the parameterized complexity class $W[2]$ [7]. If the number of variables in each clause is bounded, it is a canonical $W[1]$-complete problem [7]. These results imply that it is unlikely that these problems are *fixed-parameter tractable* (FPT). In other words, it is unlikely that they have an algorithm with running time $f(k)n^{O(1)}$ on input formulas of size n.

☆ A preliminary version of this work appeared in the Mathematical Foundations of Computer Science (MFCS) 2010, LNCS 6281, 2010, pp. 549–555.

* Corresponding author.
E-mail addresses: mail@neeldhara.com (N. Misra), swamy@cse.iitm.ac.in (N.S. Narayanaswamy), vraman@imsc.res.in (V. Raman), balsri@imsc.res.in (B.S. Shankar).

[1] Part of this work was done when the author was visiting The Institute of Mathematical Sciences, on sabbatical from IIT Madras.
[2] The *weight* of an assignment is the number of variables assigned 1 by the assignment.

0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2013.07.019

On the other hand, if the question is whether a d-CNF formula (for fixed d) has a satisfying assignment with weight *at most* k, then this generalizes the well-studied d-hitting set problem and turns out to be fixed-parameter tractable with the weight as a parameter ([17,16], see also Section 2). When we restrict our attention to 2-CNF formulas (MIN ONES 2-SAT) this problem generalizes the well-studied VERTEX COVER problem. Given a graph $G = (V, E)$, introduce a variable v for every vertex $v \in V$ and consider the formula $\bigwedge(u \vee v)$, where the $\bigwedge$ runs over all pairs u, v of variables such that (u, v) is in E. Then a satisfying assignment of weight k corresponds to a vertex cover of size k and vice versa. However, notice that we do not require negated literals to encode VERTEX COVER using 2-CNF formulas, and thus it appears that MIN ONES 2-SAT is a more general version of the vertex cover problem.

1.1. Related work

Gusfield and Pitt [8] considered this MIN ONES 2-SAT problem and gave a 2-approximation algorithm. The algorithm follows a greedy approach and gives a solution whose weight is at most twice the optimum (assuming that the formula is satisfiable). As satisfiability of 2-CNF-SAT is well known to be polynomial time solvable, we can assume without loss of generality that the given 2-SAT formula is satisfiable. See [24] for efficient exact exponential algorithms for MIN ONES 2-SAT.

One approach to design a 2-approximation algorithm for the optimization version and a $2k$-variable kernel for the parameterized version of VERTEX COVER (see Section 2 for definitions of parameterized complexity) is through linear programming. The minimum vertex cover problem can be formulated as an integer linear programming by introducing a boolean (0–1) variable x_i for every vertex $i \in V$, and by writing the constraint $x_i + x_j \geqslant 1$ $\forall (i, j) \in E$. The objective function is to minimize $\sum_{i \in V} x_i$. The linear programming relaxation of the problem relaxes the variables x_i's to take values from the interval $[0, 1]$ (instead of just values 0 and 1). A classical theorem due to Nemhauser and Trotter [20] states the following (here $n = |V|$):

Theorem 1. *(See [20].)*

- *There exists an optimum solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ to the linear programming relaxation of the integer programming formulation of* VERTEX COVER *where the variables take the values* 0, 1/2 *or* 1, *and such a solution can be found in polynomial time.*
- *There exists an optimum solution to* VERTEX COVER (*i.e. to the integer programming formulation*) *that contains all vertices i such that $x_i^* = 1$ and none of the vertices i such that $x_i^* = 0$.*

By solving the linear program, and by including all vertices with $x_i^* = 1$ into the solution and by deleting all vertices with $x_i^* = 1$ or 0, one can obtain a 2-approximation algorithm, and a $2k$-vertex kernel for VERTEX COVER. Hochbaum et al. [9] showed that the classical Nemhauser–Trotter theorem for vertex cover [20] holds for MIN ONES 2-SAT as well. This implies a 2-approximation algorithm for the optimization version, and a $2k$-variable kernel for the parameterized version of MIN ONES 2-SAT as well.

There is a reduction from 2-SAT to VERTEX COVER, pointed out by Seffi Naor (see [10]). This reduction takes an instance F of MIN ONES 2-SAT on n variables, and first computes the *closure* of the clauses (see Section 3). The closure consists of all the original clauses and clauses $(x \vee y)$ whenever $(x \vee z)$ and $(y \vee \bar{z})$ appear in the original clauses of F. From the closure of F, a graph $G(F)$ is constructed that has one vertex for every literal participating in F and an edge between a pair of literals whenever they appear together in a clause of the closure of F, and an edge $(x, \bar{x})$ for every variable x. It is shown that any satisfying assignment for F corresponds to a vertex cover of size n in $G(F)$ and conversely any vertex cover of $G(F)$ of size n corresponds to a satisfying assignment in F. However, the reduction is not 'weight preserving' in the sense that a satisfying assignment of weight k can correspond to a vertex cover of size n (recall that the graph has up to $2n$ vertices). Furthermore this reduction produces a graph with the number of vertices equal to twice the number of variables and, in the parameterized setting, does not transform k into a function of k alone. Since the reduction loses track of the weight of the solution, it does not enable us to employ VERTEX COVER to solve an instance of MIN ONES 2-SAT.

1.2. Our work

In this paper, we demonstrate a simple extension of this reduction that preserves both k and n, and allows us to carry over everything we know about VERTEX COVER to the more general setting of MIN ONES 2-SAT. Thus, we have that the apparently more general problem of MIN ONES 2-SAT can be handled as easily as vertex cover, in both the optimization and parameterized settings. In particular, the problem now has a $2k - c \lg k$-variable kernel [14,15,19] (for some constant c), a 2-approximation algorithm, and FPT and exact algorithms that will run as fast as the best algorithms for the corresponding versions of the vertex cover problem, the current best being $O^*(1.2738^k)$ [4][3] and $O^*(1.2114^n)$ [2] respectively. In particular, our reduction subsumes the earlier results (2-approximation algorithms, and Nemhauser–Trotter theorem) on this problem.

---

[3] We use the notation $O^*()$ to "hide" functions that are polynomial in the input size.