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Modularity in term rewriting revisited

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ABSTRACT

We revisit modularity in term rewriting which for the last 25 years has been a very active and fruitful research field. Starting with the pioneering works of Yoshihito Toyama on the modularity of confluence and the non-modularity of termination he thus initiated an extremely productive line of research, with many non-trivial and deep results, striking counterexamples and a substantial amount of systematic theoretical foundations, methodological principles and novel proof techniques. In this focused summary we will revisit the modularity analysis for ordinary term rewriting systems, considering various confluence and termination properties and restricting ourselves mainly to the case of disjoint unions. We will summarize known results on the (non-) modularity of various confluence and termination properties, and exhibit crucial ideas, constructions and phenomena. Later on we will also briefly consider various extensions, applications, related questions and open problems, as well as recent developments.

1. Introduction and overview

We revisit modularity in term rewriting, a field that has developed a lot since the seminal works of Yoshihito Toyama more than two decades ago. The focus will be on modularity of confluence and termination properties of (first-order) term rewriting systems (TRSs). Later we will also consider directions for extending the analysis, by weakening the disjointness requirement, looking at further properties and aspects to be investigated w.r.t. modularity, and taking into account other types of rewrite systems. We will summarize what is known so far, highlight some historic milestones, discuss basic ideas, crucial notions and concepts, main proof techniques, fundamental constructions and the main sources of difficulties in modularity analysis. Furthermore we will briefly discuss the impact of research and results in modularity on other fields and vice versa, and mention some open problems and research challenges. Since the field of modularity in (term) rewriting has become enormously diverse and rich, any such summary has to concentrate on certain aspects and questions and is thus highly incomplete and also subjective to a certain degree. In order to complete the picture a bit, we try to mention at least those aspects and subfields where modularity is also important and studied, but where due to lack of space we do not go into details. Instead, we try to give lots of pointers to the abundant literature on the respective subjects.

Let us be more concrete now and begin with something positive. In [82] Toyama proved that confluence is a modular property of disjoint TRSs whereas termination is not. The former celebrated result is fairly non-trivial, since termination of the disjoint union of two terminating systems cannot be assumed, due to the non-modularity of termination. The famous counterexample Toyama discovered is the following. [1]

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[1] In an inspiring invited talk at RTA 2005 in Nara, Japan [84] Toyama told a bit about the history of these early days of modularity. He displayed a copy of the coverpage of a manuscript where he had 'proved' the modularity of termination. When trying to complete this proof he realized that there was a non-repairable gap — and shortly afterwards he got the inspiration for the counterexample when crossing a red traffic light ….

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Example 1. Consider the two disjoint TRSs given by

$$\mathcal{R}_1 = \{f(a, b, x) \to f(x, x, x)\} \qquad \mathcal{R}_2 = \begin{Bmatrix} G(x, y) \to x \\ G(x, y) \to y \end{Bmatrix}.$$

Both systems are terminating, but their disjoint union $\mathcal{R}$ is not as witnessed by the cyclic reduction

$$f(a, b, G(a, b)) \to_{\mathcal{R}_1} f(G(a, b), G(a, b), G(a, b)) \to^{+}_{\mathcal{R}_2} f(a, b, G(a, b)) \to_{\mathcal{R}_1} \cdots .$$

This remarkably simple, beautiful and ingenious (counter)example is probably the most cited and used (combined) TRS in the literature on term rewriting since. Variations thereof have been designed and exploited in many different contexts and for many different purposes.

In our survey we will focus on modularity of properties of TRSs w.r.t. disjoint unions. Note that in general the question of whether a property of TRSs or ARSs also holds for their union and vice versa is hopeless for most interesting cases. The ARSs (or TRSs) given by $a \to b$ and $b \to a$, respectively, are both terminating, but their union is not. Similarly, $a \to b$ and $a \to c$ together yield a non-confluent system, whereas the individual one-rule systems are obviously confluent. Thus, without further restrictions the hope for obtaining non-trivial modularity results appears not justified. A drastic way of restricting the interaction between two TRSs in their union is to separate them syntactically as much as possible, by requiring that the respective signatures are disjoint. Once this case of modularity w.r.t. disjoint unions has been well understood, the disjointness requirement can be weakened, considering more general combinations of systems that are (at most) *constructor sharing*, *composable* or *hierarchically* structured. We will focus on the disjoint union case, because for most of the results and approaches for more general types of combinations developed later on the understanding, the basic ideas and proof techniques of the disjoint union case are fundamental. In some detail cases, but selectively, we will consider combinations of TRSs with shared constructors in order to illustrate which new sources of problems occur in this setting and how results from the disjoint union case can be extended by adequately taking into account these new problems.

The properties of TRSs we will investigate for their (non-)modularity are mainly confluence and termination properties, in various versions. Rather than repeating proofs from the literature we will concentrate on main ideas, crucial observations, proof techniques, fundamental constructions and analogies. Many positive modularity results have a symmetric and an asymmetric version which we will sometimes mention. This is related to a phenomenon called *vaccination* [52] where the idea is that some property becomes modular provided one of the involved TRSs in a (disjoint) union or both of them have been *vaccinated* against the danger of non-modularity, i.e. enjoy some additional property implying modularity. For the disjoint union case we will summarize the state-of-the-art concerning modularity analysis.

An interesting property similar to modularity is *persistency* (of properties) which is defined via *many-sorted* TRSs and states that a property holds for a many-sorted TRS iff it holds for the unsorted version of the system. We will also briefly summarize the most important results and open questions in this area.

Extensions of the basic modularity theory and results for more general types of combinations will only be touched, mainly for combining constructor sharing TRSs. This also applies to combinations/decompositions of systems which are not primarily specified syntactically, but using other properties resulting in certain non-disjoint combinations of systems. Nevertheless we will give numerous pointers to the relevant literature on these extensions and other versions.

Similarly, as another dimension of the modularity analysis, instead of basic TRSs also many other types of rewrite systems have been investigated for their modularity behaviour: higher-order rewrite systems (HORs), simply typed term rewriting systems (STTRSs), term graph rewriting systems, infinitary rewriting systems, conditional TRSs (CTRSs), equational term rewriting systems (i.e. where rewriting happens modulo some underlying equational theory), rewrite systems with restricted reduction relation (via strategies, contextual restrictions, forbidden patterns). We will also only briefly touch these fields, but again try to give pointers to the relevant literature.

In the paper we concentrate on confluence and termination properties, but clearly there also exist modularity results about other properties and aspects like notions related to sequentiality and neededness, complexity and constructivity. Again we only refer to the literature here.

The remainder of the paper is structured as follows. After some preliminaries in Section 2 (which we recommend the informed reader to consult only by need), in the main part (Sections 4 and 5 we present the analysis and summary of the (non-)modularity of confluence and termination properties of TRS. In the discussion in Section 7 we then reason about methodological benefits (from modularity research), applications and implications of modularity, difficulties and (some) open problems as well as recent progress and emerging trends.

Much of this summary is highly incomplete, due to the diversity, breadth and depth of the field. This applies even to the main Sections 4 and 5 where we tried to cite all relevant papers in the area. But to some degree this list of references is still certainly subjective and partially incomplete, due to the existence of papers that the author is not aware of.

2. Preliminaries

We assume familiarity with the basic theory, notions and notations of abstract and term rewriting (cf. e.g. [6,9]), and for modularity (cf. e.g. [67]). Nevertheless, for the sake of readability we repeat here some of these notions and notations, introduce acronyms for various properties and provide some basic well-known facts.