Available online at www.sciencedirect.com
SCIENCE DIRECT®
Theoretical Computer Science 355 (2006) 389–395
Theoretical Computer Science
www.elsevier.com/locate/tcs

Note

Degree-constrained decompositions of graphs: Bounded treewidth and planarity[☆]

Cristina Bazgan[a,*], Zsolt Tuza[b,c], Daniel Vanderpooten[a]

[a] LAMSADE, Université Paris-Dauphine, Place du Marechal de Lattre de Tassigny, 75775 Paris Cedex 16, France
[b] Computer and Automation Institute, Hungarian Academy of Sciences, H-1111 Budapest, Kende u. 13-17, Hungary
[c] Department of Computer Science, University of Veszprém, Hungary

Received 19 July 2004; received in revised form 13 December 2005; accepted 13 January 2006

Communicated by G. Ausiello

Abstract

We study the problem of decomposing the vertex set V of a graph into two nonempty parts V_1, V_2 which induce subgraphs where each vertex $v \in V_1$ has degree at least $a(v)$ inside V_1 and each $v \in V_2$ has degree at least $b(v)$ inside V_2. We give a polynomial-time algorithm for graphs with bounded treewidth which decides if a graph admits a decomposition, and gives such a decomposition if it exists. This result and its variants are then applied to designing polynomial-time approximation schemes for planar graphs where a decomposition does not necessarily exist but the local degree conditions should be met for as many vertices as possible.
© 2006 Elsevier B.V. All rights reserved.

Keywords: Graph decomposition; Treewidth; Planar graph; Polynomial algorithm; PTAS

1. Introduction

Given a graph $G = (V, E)$ and a subset $S \subseteq V(G)$, we denote by $d_S(v)$ the degree of a vertex $v \in V$ in $G[S]$, the subgraph of G induced by S. For $S = V$, the subscript is omitted, hence $d(v)$ stands for the degree of v in G.

Our starting point is the following general problem:

DECOMPOSITION
Input: A graph $G = (V, E)$, and two functions $a, b : V \to \mathbb{N}$ such that $a(v), b(v) \leqslant d(v)$, for all $v \in V$.
Question: Is there a nontrivial partition (V_1, V_2) of V such that $d_{V_1}(v) \geqslant a(v)$ for every $v \in V_1$ and $d_{V_2}(v) \geqslant b(v)$ for every $v \in V_2$?

☆ This research was supported by the bilateral research cooperation Balaton between EGIDE (France) and the Ministry of Education (Hungary) under Grant nos. 07244RJ and F-29/2003. The second author was also supported in part by the Hungarian Scientific Research Fund, Grant OTKA T-042710.
* Corresponding author.
E-mail addresses: bazgan@lamsade.dauphine.fr (C. Bazgan), tuza@sztaki.hu (Z. Tuza), vdp@lamsade.dauphine.fr (D. Vanderpooten).

0304-3975/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2006.01.024

A partition satisfying the above property is called a *decomposition* of G. If G admits a decomposition, we also say that G is *decomposable*. Moreover a vertex v is said to be *satisfied* when if $v \in V_1$ we have $d_{V_1}(v) \geqslant a(v)$ and if $v \in V_2$, we have $d_{V_2}(v) \geqslant b(v)$.

The decision problem DECOMPOSITION is *NP*-complete. Indeed, the rather special case where $a = b = \lceil d/2 \rceil$, introduced in [14] as SATISFACTORY PARTITION, has been shown to be *NP*-complete in [5]. Moreover, it is *NP*-complete in the range $\lceil d/2 \rceil < a = b \leqslant d - 1$, as proved in [4].

Even if the problem is *NP*-complete, polynomial instances of this problem may arise when (i) restricting the structure of the graph, or (ii) imposing constraints on a and b, or (iii) both.

Concerning case (ii), Stiebitz [21] proved that, when a and b are such that $d(v) \geqslant a(v) + b(v) + 1$ for all $v \in V$, any graph admits a decomposition. His result is not constructive. A polynomial-time algorithm that finds such a decomposition is given in [6].

In case (iii), Kaneko [19] showed that any triangle-free graph such that $d(v) \geqslant a + b$ for all $v \in V$, where a and b are positive integer constants, admits a decomposition. Diwan [12] showed that any graph with girth at least 5 such that $d(v) \geqslant a + b - 1$ for all $v \in V$, where again a and b are positive integers $\geqslant 2$ independent of v, admits a decomposition. These two results were presented for constants a and b instead of functions $a(v)$ and $b(v)$. Diwan's result was extended recently to the case of functions in [16]. However, the proofs of all these results are not constructive. In [6] we gave algorithms that find a decomposition in polynomial time for the general case of functions, provided that their sum (or sum minus 1) does not exceed the degree function.

In this paper we study DECOMPOSITION in case (i), i.e. without any restrictions on the functions a, b (apart from the trivial one that they should not exceed the degree function d), but imposing restrictions on the class of graphs. We are not aware of any previous result concerning this case. We show here that, for graphs with bounded treewidth, one can decide in polynomial time if a graph is decomposable, and give in polynomial time a decomposition when it exists.

It should be noted that the general result developed by Courcelle [11]—stating that any problem expressible in second-order monadic logic is polynomial-time solvable for graphs of bounded treewidth—cannot be applied directly here. The technical difficulty concerning the applicability of this result for DECOMPOSITION is that if $a(v)$ and $b(v)$ cannot be expressed for almost all $v \in V$ in a uniform way (e.g., $a(v) = a$ and $b(v) = b$ are constants, or $d(v)/a(v)$ and $d(v)/b(v)$ are independent of v), then the formula for the problem in second-order monadic logic is not of constant length. A similar technical complication yields the nonapplicability of Gerber and Kobler's result [15] on graphs of bounded clique-width.

We also study some variants of DECOMPOSITION with additional constraints on the size of the vertex classes. Let $t = t(n)$ be an integer-valued function such that $0 \leqslant t(n) \leqslant n$ for every $n \in \mathbb{N}$. We consider the following problem:

t-DECOMPOSITION
Input: A graph $G = (V, E)$, and two functions $a, b : V \to \mathbb{N}$ such that $a(v), b(v) \leqslant d(v)$, for all $v \in V$.
Question: Is there a partition (V_1, V_2) of V with $|V_1| = t(|V|)$ such that $d_{V_1}(v) \geqslant a(v)$ for every $v \in V_1$ and $d_{V_2}(v) \geqslant b(v)$ for every $v \in V_2$?

A partition (V_1, V_2) such that $|V_1| = t(|V|)$ is called a *t-partition*. A *t*-partition where all vertices are satisfied is called a *t-decomposition*. If G admits a *t*-decomposition, we also say that G is *t-decomposable*.

A particular interesting case of this problem is when the graph has an even number of vertices and we consider only balanced partitions ($t = n/2$, where n is the number of vertices), giving rise to the following problem:

BALANCED DECOMPOSITION
Input: A graph $G = (V, E)$ with an even number of vertices, and two functions $a, b : V \to \mathbb{N}$ such that $a(v), b(v) \leqslant d(v)$, for all $v \in V$.
Question: Is there a partition (V_1, V_2) of V with $|V_1| = |V_2|$ such that $d_{V_1}(v) \geqslant a(v)$ for every $v \in V_1$ and $d_{V_2}(v) \geqslant b(v)$ for every $v \in V_2$?

Since an input graph may not have any (t-)decomposition, it is of interest to study the corresponding *optimization* problem where we try to satisfy as many of the vertices as possible. (In this setting it is not necessary to assume anymore that the vertex degree $d(v)$ is an upper bound on $a(v)$ and $b(v)$.) We consider then the two following

Download English Version:

https://daneshyari.com/en/article/438856

Download Persian Version:

https://daneshyari.com/article/438856

Daneshyari.com