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Drawing graphs with right angle crossings☆

Walter Didimo [a,*], Peter Eades [b], Giuseppe Liotta [a]

[a] *Dip. di Ingegneria Elettronica e dell'Informazione, Università degli Studi di Perugia, Italy*

[b] *Department of Information Technology, University of Sydney, Australia*

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ABSTRACT

Cognitive experiments show that humans can read graph drawings in which all edge crossings are at right angles equally well as they can read planar drawings; they also show that the readability of a drawing is heavily affected by the number of bends along the edges. A graph visualization whose edges can only cross perpendicularly is called a *RAC (Right Angle Crossing) drawing*. This paper initiates the study of combinatorial and algorithmic questions related to the problem of computing RAC drawings with few bends per edge. Namely, we study the interplay between number of bends per edge and total number of edges in RAC drawings. We establish upper and lower bounds on these quantities by considering two classical graph drawing scenarios: The one where the algorithm can choose the combinatorial embedding of the input graph and the one where this embedding is fixed.

1. Introduction

The problem of making good drawings of relational data sets is fundamental in several applications. To enhance human understanding the drawing must be readable, that is it must easily convey the structure of the data and of their relationships (see, for example, [10,26,27]).

A tangled rat's nest of a diagram can be confusing rather than helpful. Intuitively, one may measure the "tangledness" of a graph layout by the number of its edge crossings and by the number of its bends along the edges. This intuition has some scientific validity: Experiments by Purchase et al. have shown that performance of humans in path tracing tasks is negatively correlated to the number of edge crossings and to the number of bends in the drawing [34,35,40].

This negative correlation has motivated intense research about how to draw a graph with few edge crossings and small *curve complexity* (i.e., maximum number of bends along an edge). As a notable example we recall the many fundamental combinatorial and algorithmic results about planar or quasi-planar straight-line drawings of graphs (see, for example, [29,30]). However, in many practical cases the relational data sets do not induce planar or quasi-planar graphs and a high number of edge crossings is basically not avoidable, especially when a particular drawing convention is adopted. How to handle these crossings in the drawing remains unanswered.

Recent cognitive experiments of network visualization provide new insights into the classical correlation between edge crossings and human understanding of a network visualization. Huang et al. show that the edge crossings do not inhibit human task performance if the edges cross at a large angle [22,23,25]. In fact, professional graphic artists commonly use large crossing angles in network drawings. For example, crossings in hand drawn metro maps and circuit schematics

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* Corresponding author. Tel.: +39 075 5853680; fax: +39 075 5853654.
E-mail addresses: didimo@diei.unipg.it (W. Didimo), peter@cs.usyd.edu.au (P. Eades), liotta@diei.unipg.it (G. Liotta).

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are conventionally at 90° (see, for example, [39]). Also, in the guidelines of the CCITT (Comité Consultatif International Téléphonique et Télégraphique) for drawing Petri nets the following requirement is reported: "There should be no acute angles where arcs cross" [9].

This paper initiates the study of combinatorial and algorithmic questions related to the problem of computing drawings of graphs where the edges cross at 90°. Graph visualizations of this type are called *RAC (Right Angle Crossing) drawings*. We study the interplay between the curve complexity and total number of edges in RAC drawings and establish upper and lower bounds on these quantities. It is immediate to see that every graph has a RAC drawing where the edges are represented as simple Jordan curves that are "locally adjusted" around the crossings so that they are orthogonal at their intersection points. However, not every graph has a RAC drawing if small curve complexity is required.

We consider two classical graph drawing scenarios: In the *variable embedding setting* the drawing algorithm takes as input a graph G and attempts to compute a RAC drawing of G; the algorithm can choose both the circular ordering of the edges around the vertices and the sequence of crossings along each edge. In the *fixed embedding setting* the input graph G is given along with a fixed ordering of the edges around its vertices and a fixed ordering of the crossings along each edge; the algorithm must compute a RAC drawing of G that preserves these fixed orderings. An outline of our results is as follows.

- In Section 3 we study the combinatorial properties of straight-line RAC drawings in the variable embedding setting. We give a tight upper bound on the number of edges of straight-line RAC drawings. Namely, we prove that straight-line RAC drawings with n vertices can have at most $4n - 10$ edges, and that there exist infinitely many graphs with this number of edges that are straight-line RAC drawable. It might be worth recalling that straight-line RAC drawings are a subset of the quasi-planar drawings, for which the problem of finding a tight upper bound on the edge density is still open (see, for example, [2,3,32]).
- Motivated by the previous result, we study in Section 4 how the edge density of RAC drawable graphs varies with the curve complexity. We show how to compute a RAC drawing whose curve complexity is three for any graph in the variable embedding setting. We also show that this bound on the curve complexity is tight by proving that curve complexity one implies $O(n^{\frac{4}{3}})$ edges and that curve complexity two implies $O(n^{\frac{7}{4}})$ edges.
- In Section 5 we investigate the fixed embedding setting. In contrast with the results for the variable embedding setting, we show that in the fixed embedding setting the curve complexity of a RAC drawing may no longer be constant. Namely, we establish an $\Omega(n^2)$ lower bound on the curve complexity in this scenario. The embedded graphs constructed for establishing this bound have the properties that the number of crossings between any two edges is bounded by a constant (independent of n). We also show that if any two edges cross at most k times, it is always possible to compute a RAC drawing with $O(kn^2)$ curve complexity. This last result implies that the quadratic bound in n is tight if we restrict to those embeddings such that the number of crossings between any two edges is bounded by a constant.

Preliminary definitions and basic properties of RAC drawings are given Section 2. Conclusions and open problems can be found in Section 6.

We remark that, right after the ideas of this paper were disseminated (they have been mentioned in an invited talk by Eades at ISAAC 2008 [18], informally communicated to the attendees of the Bertinoro Workshop on Graph Drawing 2009,[1] and presented at WADS 2009 [13]), the study of RAC drawings and of its variants has been receiving increasing interest. Angelini et al. study upward RAC drawings and specific sub-families of non-planar graphs that are RAC drawable with few bends per edge [4]. Arikushi et al. prove linear upper bounds to the number of edges of poly-line RAC drawings [7], thus improving the sub-quadratic bounds presented in this paper. RAC drawings of bipartite graphs are studied in [11,14]. Argyriou et al. prove that deciding whether a graph G admits a straight-line RAC drawing is $\mathcal{NP}$-hard [6]. Relationships between straight-line RAC drawings and 1-planar drawings are described in [20]. The advantages of drawing planar graphs with right angle crossings are investigated by van Kreveld [38]. Relaxations of RAC drawings have been studied by several authors. Namely, drawings where edge crossings form angles of at least α (for some fixed constant $0 < \alpha < \pi/2$), have been independently studied by Di Giacomo et al. [12] and by Dujimović et al. [17]. For this kind of drawing, Di Giacomo et al. prove bounds and trade-offs on the area requirement and number of bends, while Dujimović et al. give bounds on the number of edges. In [17] an alternative proof of the upper bound $4n - 10$ to the number of edges of a straight-line RAC drawing is also presented; this proof is based on charging techniques and on a case analysis similar to the one used in this paper. Drawings where edge crossings form angles of exactly α (for some fixed constant $0 < \alpha < \pi/2$) are studied by Ackerman et al. [1]; they prove that these drawings always have a linear number of edges. Finally, algorithms and systems for computing drawings with good crossing angle resolution are described in [5,15,16,19,24,28].

2. Preliminaries

We recall some basic definitions about graph drawing and graph planarity. For more details see [10]. Let G be a graph. A *drawing* of G is a geometric representation of G in the plane such that each vertex is drawn as a distinct point of the plane and each edge is drawn as a simple Jordan curve between the points representing its end-vertices. A *poly-line drawing* of

[1] http://www.diei.unipg.it/~bwgd09/.