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On the hardness of finding near-optimal multicuts in directed acyclic graphs ☆

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ABSTRACT

Given an edge-weighted (di)graph and a list of source–sink pairs of vertices of this graph, the *minimum multicut problem* consists in selecting a minimum-weight set of edges (or arcs), whose removal leaves no path from each source to the corresponding sink. This is a well-known $\mathcal{NP}$-hard problem, and improving several previous results, we show that it remains $\mathcal{APX}$-hard in unweighted directed acyclic graphs (DAG), even with only two source–sink pairs. This is also true if we remove vertices instead of arcs.

1. Introduction

In this paper, we study the *minimum multicut problem* (MinMC), which is a classical problem in graph theory and combinatorial optimization [9].

Assume that we are given an n-vertex m-edge (di)graph $G = (V, E)$, a *weight function* $w : E \to \mathbb{N}^*$ and a list $\mathcal{N}$ of pairs (source s_i, sink s'_i) of *terminal* vertices of G. A *multicut* is a set of edges (or arcs) of G, whose removal leaves no (directed) path from s_i to s'_i for each i. The *weight* of a multicut is the sum of the weights of its edges (or arcs). MinMC consists in computing a minimum-weight multicut. The graph G is *unweighted* if $w(e) = 1$ for each edge (or arc) $e \in E$. The *minimum multiterminal cut problem* (MinMTC) is a particular minimum multicut problem in which, given a set of r vertices $\{t_1, \ldots, t_r\}$, the source–sink pairs are (t_i, t_j) for $i \neq j$.

We also need to introduce some notions from approximation theory: given a minimization (resp. maximization) problem Π and a real $\alpha \geq 1$, an *α-approximation algorithm* for Π is an algorithm that runs in polynomial time and outputs a feasible solution whose value is at most α times the value of an optimal solution for Π (resp. whose value multiplied by α is at least the value of an optimal solution for Π). A *polynomial-time approximation scheme* (PTAS) is an algorithm that, for any given $\epsilon > 0$, is an $(1 + \epsilon)$-approximation algorithm. A problem is *$\mathcal{APX}$-hard* iff it does not admit a PTAS unless $\mathcal{P} = \mathcal{NP}$. A problem is *$\mathcal{APX}$-complete* if it is $\mathcal{APX}$-hard and admits an α-approximation algorithm for some fixed real $\alpha \geq 1$.

For $|\mathcal{N}| = 1$, MinMC is equivalent to the *minimum cut problem*, and therefore is polynomial-time solvable both in directed and in undirected graphs [9]. For the same reason, MinMTC is polynomial-time solvable in undirected graphs when $r = 2$. However, MinMTC becomes $\mathcal{NP}$-hard, and even $\mathcal{APX}$-hard, as soon as $r = 3$ in undirected graphs [10], and as soon as $r = 2$ in digraphs [14]. As a consequence, MinMC is $\mathcal{APX}$-hard in digraphs for each fixed $|\mathcal{N}| \geq 2$, and in undirected graphs for each fixed $|\mathcal{N}| \geq 3$ (the case $|\mathcal{N}| = 2$ being polynomial-time solvable [23]). For an arbitrary number of source–sink pairs, MinMC is $\mathcal{APX}$-hard even in unweighted stars [13]. Moreover, MinMC is polynomial-time solvable in directed

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trees (the constraint matrix being totally unimodular in this case) and MinMTC is polynomial-time solvable in directed acyclic graphs (DAGs) [2,9]. (Recall that a *circuit* of a digraph is a directed cycle, i.e., a directed path where all arcs share the same orientation and whose endpoints coincide, and that a *directed acyclic graph* is a digraph without circuits.) Finally, there is a polynomial-time approximation scheme (PTAS) for MinMC in unweighted graphs of bounded tree-width and bounded degree, but dropping any of these three assumptions leads to $\mathcal{APX}$-hardness (instead of $\mathcal{NP}$-hardness only) [6]. This PTAS also holds for a variant of MinMC in digraphs, called MinMC-SC, where the goal is to remove a minimum-weight set of arcs so that, for each i, no directed cycle (circuit) contains both s_i and s'_i.

A related problem is the well-known *maximum multicommodity flow problem* (MaxMF): indeed, these problems can be modeled by linear programs (LP), whose continuous relaxations are dual [9]. Hence, the optimal value of MaxMF is smaller than or equal to the one of MinMC, and the optimal values of their continuous relaxations are equal. However, unlike MinMC, MaxMF is $\mathcal{NP}$-hard in undirected graphs when $|\mathcal{N}| = 2$ [12]. Note that, for both problems, the (general) directed case is harder than the undirected case: indeed, given an undirected instance, one can obtain an equivalent directed instance by replacing each edge (u, v) of weight $w(u, v)$ by a gadget consisting of five arcs (u, w_{uv}), (v, w_{uv}), (w'_{uv}, u), (w'_{uv}, v), (w_{uv}, w'_{uv}) of weight $w(u, v)$, where w_{uv} and w'_{uv} are two new vertices.

Some properties for MinMC are given in [16], where the authors conjecture that this problem is not significantly simpler in directed acyclic graphs. In [3], it was proved that MinMC is $\mathcal{NP}$-hard even in unweighted directed acyclic graphs having a very special structure (namely, the underlying undirected graph is a bipartite cactus of bounded path-width and of maximum degree three), and $\mathcal{APX}$-hard in unweighted digraphs of bounded maximum degree and bounded directed tree-width (see [20] for a definition). Moreover, for fixed $|\mathcal{N}|$, an algorithm solving MinMC in polynomial time in a class of DAGs was presented. However, for fixed $|\mathcal{N}|$, the complexity of MinMC in general DAGs is still open, while the related problem MaxMF is known to be $\mathcal{NP}$-hard for a long time [12].

In this paper, we show that this problem is $\mathcal{APX}$-hard, even if $|\mathcal{N}| = 2$. Before proving this result in Section 3, we first consider MaxMF and give in Section 2 a construction generalizing the one described in [19]: we show that the integrality gap for MaxMF can be arbitrarily close to 2 in DAGs with $|\mathcal{N}| = 2$, and that, for any integer $p > 1$, there exist instances of size polynomial in p such that the minimum value of a fractional multicut (and hence the maximum value of a fractional multiflow) in these graphs is equal to a multiple of $1/p$. Finally, we give in Section 4 a second reduction, that is valid only for $|\mathcal{N}| \geq 3$, but in which all the non-terminal vertices have maximum degree three and all the arcs have weight 1.

2. An infinite family of instances for MaxMF

The continuous relaxation of MinMC consists in labeling each arc $e \in E$ with a value $x(e)$, while minimizing $\sum_{e\in E} w(e)x(e)$ and ensuring that, for each i and for each path p_j from s_i to s'_i, we have $\sum_{e\in p_j} x(e) \geq 1$. MaxMF consists in routing the maximum number of flow units between the source–sink pairs, while ensuring that the total flow on each arc does not exceed its weight. It is known that, when the graph is undirected and $|\mathcal{N}| = 2$, the continuous relaxations of both MinMC and MaxMF have optimal solutions in which variables are multiples of $1/2$: this property allows to solve MaxMF in polynomial time in this case, if all the capacities are even [17,22]. One could hope that, if such a property holds when the graph is a DAG and $|\mathcal{N}| = 2$, then this might help to solve the problem efficiently, at least in special cases. Itai showed in [19] that unfortunately this is not the case: more precisely, he gave an example showing that the variables of an optimal solution can all be multiples of $1/3$, and where the integrality gap for MaxMF is $5/3$. Here, we generalize this construction.

More precisely, we show that, in DAGs with $|\mathcal{N}| = 2$, on the one hand the integrality gap for MaxMF can be arbitrarily close to 2, and, on the other hand, for any integer $p > 1$, there exist instances of size polynomial in p in which the value of an optimal fractional multiflow and the value of each variable in this optimal solution is equal to some multiple of $1/p$. This implies, on the one hand, that no approximation algorithm with a ratio better than 2 can be obtained from an optimal fractional multiflow, and, on the other hand, that scaling the weights by some constant is not sufficient to ensure an integer optimal solution to the continuous relaxation of MaxMF. Note that, when $|\mathcal{N}| = 2$, a trivial 2-approximation algorithm for MaxMF is obtained by computing a maximum flow between s_1 and s'_1 and a maximum flow between s_2 and s'_2, and then keeping the best one.

Given $p > 1$, we construct the following instance I_p: we start from a path with $2p$ vertices $v_1, \ldots, v_{2p}$ and orient its arcs from v_1 to v_{2p}. Then, we define the first terminal pair (s_1, s'_1) by setting $s_1 = v_1$ and $s'_1 = v_{2p}$. We introduce the second terminal pair (s_2, s'_2): s_2 is linked to a new vertex u by a unique arc (s_2, u), and, for each $i \in \{1, \ldots, p\}$, vertex u is linked to v_{2i-1} by an arc (u, v_{2i-1}). Finally, for each $i \in \{1, \ldots, p\}$, vertex v_{2i} is linked to s'_2 by an arc (v_{2i}, s'_2), and the $4p$ arcs of the obtained DAG have weight 1. In I_p, the optimal solutions for MinMC and MaxMF are then easy to compute. On the one hand, the optimal solution for MinMC has weight 2, and is obtained by removing the arcs (s_2, u) and (s_1, v_2). On the other hand, an optimal solution for MaxMF has value 1, and is obtained by routing one unit of flow between s_1 and s'_1 (or between s_2 and s'_2). The following lemma deals with the continuous relaxations of MinMC and MaxMF in I_p:

Lemma 1. *In I_p, the optimal solutions of the continuous relaxations of* MinMC *and* MaxMF *have value* $2 - \frac{1}{p}$.

Proof. In I_p, the optimal solution of the continuous relaxation of MinMC is obtained by setting $x(s_2, u) = \frac{p-1}{p}$ and $x(v_{2i-1}, v_{2i}) = \frac{1}{p}$ for each $i \in \{1, \ldots, p\}$: this is indeed a solution, since the path from s_1 to s'_1 uses the p arcs (v_{2i-1}, v_{2i}), $i \in$