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# Corrected upper bounds for free-cut elimination

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ABSTRACT

Free-cut elimination allows cut elimination to be carried out in the presence of non-logical axioms. Formulas in a proof are anchored provided they originate in a non-logical axiom or non-logical inference. This paper corrects and strengthens earlier upper bounds on the size of free-cut elimination. The correction requires that the notion of a free cut be modified so that a cut formula is anchored provided that all of its introductions are anchored, instead of only requiring that one of its introductions is anchored. With the correction, the originally proved size upper bounds remain unchanged. These results also apply to partial cut elimination. We also apply these bounds to elimination of cuts in propositional logic.

If the non-logical inferences are closed under cut and infer only atomic formulas, then all cuts can be eliminated. This extends earlier results of Takeuti and of Negri and von Plato.


## 1. Introduction

The notion of *free-cut elimination* was introduced by Takeuti [12] as an extension of cut elimination that can be used in the presence of induction inference rules. In short, the free-cut elimination theorem states that any provable sequent can be proved using only cuts in which at least one of the cut formulas was introduced as a principal formula of an induction axiom. However, Takeuti did not provide a detailed proof of the free-cut elimination.

Free-cut elimination is important for results about computational complexity or constructivity in proof theory. For instance, the second author used free-cut elimination for witnessing theorems in bounded arithmetic [3], and many other researchers have used it for similar purposes.

A different version of free-cut elimination was later used by the second author in the expository article [5]. In this variant, a set $\mathfrak{S}$ of non-logical axioms is allowed, and any formula that occurs in a non-logical axiom is called *anchored*. Cuts in which neither cut formula is anchored are called *free*, and the modified free-cut elimination theorem states that any provable sequent is provable by a proof in which no cuts are free.

However, as William Scott [private communication] first pointed out, there is an error in the proof of the free-cut elimination theorem in [5]. As a result, although the free-cut elimination theorem is indeed correct, the upper bounds on the size of free-cut free proofs that are obtained in [5] are not correct as stated.

Part of the goal of the present paper is to correct this. The fix does not involve changing the upper bounds themselves, rather it involves changing the definition of anchored and free formulas, as well as the definition of the *depth* of a cut formula. In fact, the revised theorem proved in the present paper is stronger than the result proved in [5], since the new definition of anchored is stricter than the original definition. The basic difference in the two notions of anchored is that the original definition specified that a formula is anchored if at least one of the places it is introduced is an anchor, whereas the revised definition requires that every place the formula is introduced be an anchor.

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At the same time, we will prove the free-cut elimination theorem in a somewhat more general setting, by allowing a more general notion of non-logical initial sequents and non-logical rules. This unifies the two notions of free-cut elimination from [12] and [5].

For propositional logic, this gives a proof that non-atomic free cuts can be eliminated with only an exponential blowup in the size of proofs. This generalizes results of Zhang [14] and Gerhardy [6] by showing that these bounds apply even in the presence of non-logical axioms when eliminating free cuts.

Section 6 proves theorems about when cuts can be completely eliminated even in the presence of non-logical axioms and inferences. This generalizes work of Takeuti on generalized equality axioms, as well as the non-logical rules of inference used by Negri and von Plato to simulate arbitrary quantifier-free (i.e., purely universal) axioms.

It should be stressed that the free-cut elimination theorems stated in prior works [12,5] are correct as stated, with the sole exception of the upper bounds in [5]. Fortunately, it seems that the applications of anchored cuts and formulas depth as defined in [5] have been used only in ways that have not generated further errors. This is because those upper bounds have been used only for common systems, not for contrived systems. Indeed, the results and upper bounds as stated in [5] *are* correct for all commonly used systems such as $I\Sigma_k$, $S^k_2$, $T^k_2$, etc., because of the special nature of the induction axioms. Section 5 proves results about partial cut elimination, and these results seemingly cover all existing applications of free-cut elimination to-date.

Our proofs will all use "global" transformations of proofs in the style of the proof of cut elimination in [5]. It would also be possible to prove the theorems using induction on the height of proofs, by using reductions that act on the final inferences of proofs as was done by Gentzen in the original proofs of cut elimination. Indeed, induction on the height of proofs is the most common way to carry out the proofs and is used by many authors, see for instance in the proofs by [10,14,6,13] who obtain bounds very similar to those of the present paper. An advantage to our global proof method is that it makes more explicit how proofs are transformed for cut elimination.

A rather different approach to cut elimination is given by Baaz and Leitsch [1,2], who reduce cut elimination to resolution. In some special cases they obtain super-elementarily better upper bounds on the size of cut free proofs than can be obtained by Gentzen reduction methods, but they do not give the same kind of tight bounds for general cut elimination as the present paper.

## 2. The sequent calculus and free cuts

We presume the reader has basic familiarity with the sequent calculus and cut elimination, but begin by reviewing the necessary definitions for the systems used later in the paper. We work with a sequent calculus for classical logic over the connectives $\forall$, $\exists$, $\land$, $\lor$, $\supset$, and $\neg$. Lines in a sequent calculus proof are called *sequents* and have the form $\Gamma \to \Delta$, where the *cedents* $\Gamma$ and $\Delta$ are finite sequences of formulas. The *logical initial sequents* are $A \to A$, with $A$ required to be an atomic formula. The valid *logical inferences* are as shown below.

$$\text{Exchange:left}\ \frac{\Gamma, A, B, \Lambda \to \Delta}{\Gamma, B, A, \Lambda \to \Delta} \qquad \text{Exchange:right}\ \frac{\Gamma \to \Delta, A, B, \Lambda}{\Gamma \to \Delta, B, A, \Lambda}$$

$$\text{Contraction:left}\ \frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta} \qquad \text{Contraction:right}\ \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}$$

$$\text{Weakening:left}\ \frac{\Gamma \to \Delta}{A, \Gamma \to \Delta} \qquad \text{Weakening:right}\ \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}$$

$$\neg\text{:left}\ \frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta} \qquad \neg\text{:right}\ \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A}$$

$$\land\text{:left}\ \frac{A, B, \Gamma \to \Delta}{A \land B, \Gamma \to \Delta} \qquad \land\text{:right}\ \frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \land B}$$

$$\lor\text{:left}\ \frac{A, \Gamma \to \Delta \qquad B, \Gamma \to \Delta}{A \lor B, \Gamma \to \Delta} \qquad \lor\text{:right}\ \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \lor B}$$

$$\supset\text{:left}\ \frac{\Gamma \to \Delta, A \qquad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta} \qquad \supset\text{:right}\ \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}$$

$$\forall\text{:left}\ \frac{A(t), \Gamma \to \Delta}{(\forall x)A(x), \Gamma \to \Delta} \qquad \forall\text{:right}\ \frac{\Gamma \to \Delta, A(b)}{\Gamma \to \Delta, (\forall x)A(x)}$$

$$\exists\text{:left}\ \frac{A(b), \Gamma \to \Delta}{(\exists x)A(x), \Gamma \to \Delta} \qquad \exists\text{:right}\ \frac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, (\exists x)A(x)}$$

$$\text{Cut}\ \frac{\Gamma \to \Delta, A \qquad A, \Gamma \to \Delta}{\Gamma \to \Delta}$$