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Cops and Robbers from a distance

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ABSTRACT

Cops and Robbers is a pursuit and evasion game played on graphs that has received much attention. We consider an extension of Cops and Robbers, distance k Cops and Robbers, where the cops win if at least one of them is of distance at most k from the robber in G. The cop number of a graph G is the minimum number of cops needed to capture the robber in G. The distance k analogue of the cop number, written $c_k(G)$, equals the minimum number of cops needed to win at a given distance k. We study the parameter c_k from algorithmic, structural, and probabilistic perspectives. We supply a classification result for graphs with bounded $c_k(G)$ values and develop an $O(n^{2s+3})$ algorithm for determining if $c_k(G) \leq s$ for s fixed. We prove that if s is not fixed, then computing $c_k(G)$ is NP-hard. Upper and lower bounds are found for $c_k(G)$ in terms of the order of G. We prove that

$$\left(\frac{n}{k}\right)^{1/2+o(1)} \leq c_k(n) = O\left(\frac{n}{\log\left(\frac{2n}{k+1}\right)}\frac{\log(k+2)}{k+1}\right),$$

where $c_k(n)$ is the maximum of $c_k(G)$ over all n-vertex connected graphs. The parameter $c_k(G)$ is investigated asymptotically in random graphs $G(n, p)$ for a wide range of $p = p(n)$. For each $k \geq 0$, it is shown that $c_k(G)$ as a function of the average degree $d(n) = pn$ forms an intriguing zigzag shape.

1. Introduction and main results

Originating with the work of Nowakowski and Winkler [25], Quilliot [26], and Aigner and Fromme [1] in the 1980's on the game of Cops and Robbers, a large and diverse corpus of research has now emerged on pursuit and evasion games on graphs. In pursuit and evasion games, the usual setting is a discrete-time two-person game consisting of an intruder who is loose on the vertices of a graph and trying to evade capture, and a set of searchers whose goal is to capture the robber while minimizing resources. Networks that require a smaller number of searchers may be viewed as more secure than those where many searchers are needed. Variations allow for players to possess only imperfect information, utilize only certain types of movements, allowing the players to move at various speeds, or meet specified conditions to win the game. See [12] for a survey of such variations. For example, as is the case in this work, a searcher need not occupy the vertex of the robber to capture him, but must "see" or "shoot" the robber from some prescribed distance away. For analogies from computer gaming, classic Cops and Robbers is akin to a moving-target game where the intruder must be touched to lose (such as Pac-Man), while the scenarios we consider compare with first-person shooter games where weapons hit targets at some prescribed distance. For recent surveys on pursuit and evasion games, the reader is directed to [2,12,16].

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We give a formal description of the game of distance k Cops and Robbers, by first recalling how Cops and Robbers is played. In Cops and Robbers, there are two players, a set of s *cops* (or *searchers*) $\mathcal{C}$, where $s > 0$ is a fixed integer, and the *robber* $\mathcal{R}$. The cops begin the game by occupying a set of s vertices of an undirected, and finite graph G. We take G to be *reflexive*: there are loops on each vertex. While the game may be played on a disconnected graph, without loss of generality, assume that G is connected (since the game is played independently on each component and the number of cops required is the sum over all components). The cops and robber move in *rounds* indexed by nonnegative integers. Each round consists of movements by one or more cops, followed by a move by the robber. More than one cop is allowed to occupy a vertex, and the players may *pass*; that is, remain on their current vertices. A *move* in a given round for a cop or the robber consists of a pass or moving to an adjacent vertex; each cop may move or pass in a round. The players know each other's current locations; that is, the game is played with *perfect information*. The cops win and the game ends if at least one of the cops can eventually occupy the same vertex as the robber; otherwise, $\mathcal{R}$ wins. Note that if s cops win the game so that in round 0 they occupy a set of vertices S, then they may win by occupying any set of vertices in round 0 (simply move the cops to the vertices of S, and then play as if starting the game at S). As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c(G)$, which is the minimum cardinality of the set of cops needed to win on G. While this vertex pursuit game played with one cop was introduced in [25,26], the cop number was first introduced in [1].

We study a variation of the game of Cops and Robbers in which cops have the ability of catching the robber if he is sufficiently close. More precisely, fix a nonnegative integer parameter k. The game of *distance k Cops and Robbers* is played in a way analogous to Cops and Robbers, except that the cops win if a cop is within distance at most k from the robber (for simplicity, we identify the players with the vertices they occupy). If $k = 0$, then distance k Cops and Robbers reduces to the classical Cops and Robbers game.

The minimum number of cops which possess a winning strategy in G playing distance k Cops and Robbers is denoted by $c_k(G)$. Hence, $c_0(G)$ is just the usual cop number $c(G)$. For example, for the 4-cycle, $c_0(C_4) = 2$, while $c_k(C_4) = 1$ for all $k \geq 1$. Note that for G connected, $c_k(G) = 1$ if $k \geq \text{diam}(G) - 1$, where $\text{diam}(G)$ is the diameter of G. Further, for all $k \geq 1$, $c_k(G) \leq c_{k-1}(G)$.

We observe that for given integers $k, m \geq 1$, there are examples of graphs with the property that $c_k(G) = 1$ but $c(G) = m$. To see this, we consider random graphs. The *random graph* $G(n, p)$ consists of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all graphs with vertex set $[n] = \{1, 2, \ldots, n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and for every $G \in \Omega$

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

This space may be viewed as $\binom{n}{2}$ independent coin flips, one for each pair of vertices, where the probability of success (that is, drawing an edge) is equal to p. Note that $p = p(n)$ can tend to zero with n. We say that an event holds *asymptotically almost surely* (a.a.s.) if it holds with probability tending to 1 as $n \to \infty$. Now, if $p \in (0, 1)$ is constant, then the random graph $G(n, p)$ a.a.s. satisfies $c(G(n, p)) = \Theta(\log n)$ (see [7]), but a.a.s. $c_k(G(n, p)) = 1$ for all $k > 0$ since a.a.s. it has diameter 2.

In the case $k = 0$, polynomial-time algorithms were given in [4,15,17] for recognizing if G satisfies $c_0(G) \leq s$, where s is a fixed positive integer. In particular, it is implicit in the work of [17] that their algorithm runs in time $O(n^{2s+3})$, where $n = |V(G)|$.

A difficult open problem in graph searching is Meyniel's conjecture (communicated by Frankl [13]), which states that $c_0(G) = O(\sqrt{n})$. Up until recently, the best known upper bound for general graphs was given in [9] where it was proved that $c_0(n) = O(\frac{n}{\log n})$. Recent work of from [14,23,28] proved using the probabilistic method that $c_0(n) = O\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2(n)}}}\right)$. Meyniel's conjecture has been essentially verified for $G(n, p)$ random graphs for several cases when p is a function of n; see [6–8,24].

We study the parameter c_k from algorithmic, structural, and probabilistic perspectives. In particular, we consider both algorithms and bounds for $c_k(G)$, as well as the game played on $G(n, p)$. In Section 2, we analyze the complexity of computing $c_k(G)$ for a given graph G. We give a polynomial-time algorithm for determining whether $c_k(G)$ is equal to s, assuming that s is fixed. Our algorithm runs in time $O(n^{2s+3})$ (see Theorem 3), regardless of the value of k. For any two integers s and k, Theorem 1 gives a classification of the family of graphs with $c_k(G) > s$ using the strong product of graphs. Despite Theorem 3, we prove in Corollary 10 that for any integer $k \geq 0$ there is no polynomial-time algorithm to compute $c_k(G)$, unless P = NP.

In Sections 3 and 4, we supply upper and lower bounds for $c_k(G)$ in terms of the order of G; see Theorems 4 and 11, respectively. We let $c_k(n)$ denote the maximum of $c_k(G)$ over all n-vertex connected graphs. It is shown that

$$\left(\frac{n}{k}\right)^{1/2+o(1)} \leq c_k(n) = O\left(\frac{n}{\log\left(\frac{2n}{k+1}\right)} \frac{\log(k+2)}{k+1}\right).$$

These bounds generalize known bounds for the cop number, but require new techniques which are of interest in their own right. In Theorem 12, we present asymptotic results for $c_k(G(n, p))$, where $p = p(n)$. In particular, for each $k \geq 0$, the graph of the function $c_k(G(n, p))$ follows a characteristic zigzag shape (see Fig. 3). Theorem 12 and the results of Section 5 generalize the results of [24] which considered the case $k = 0$.

All graphs we consider are undirected, finite, connected, and reflexive (that is, all vertices contain one loop), unless otherwise stated. The kth *closed neighborhood of a vertex* x in G, written $N_G{}^k[x]$, consists of all vertices of distance at most k from x in G, including the vertex x itself; in the case $k = 1$, we write simply $N_G[x]$. The kth *closed neighborhood of a set*