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Valiant's Holant Theorem and matchgate tensors

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Abstract

We propose *matchgate tensors* as a natural and proper language to develop Valiant's new theory of Holographic Algorithms. We give a treatment of the central theorem in this theory – the Holant Theorem – in terms of matchgate tensors. Some generalizations are presented.

Keywords: Holographic algorithms; Perfect matchings; Matchgates; Matchcircuit; Matchgrid; Signatures; Grassmann–Plücker identities; Covariant and contravariant tensors; Holant

1. Background

In a remarkable paper, Valiant [17] in 2004 has proposed a completely new theory of Holographic Algorithms or Holographic Reductions. In this framework, Valiant has developed a most novel methodology of designing polynomial time (indeed NC^2) algorithms, a methodology by which one can design a custom made process capable of carrying out a seemingly exponential computation with exponentially many cancellations so that the computation can actually be done in polynomial time.

The simplest analogy is perhaps with Strassen's matrix multiplication algorithm [11]. Here the algorithm computes some extraneous quantities in terms of the submatrices, which do not directly appear in the answer yet only to be canceled later, but the purpose of which is to speed up computation by introducing cancellations. In the several cases where such clever algorithms had been found, they tend to work in a linear algebraic setting, in particular the computation of the determinant figures prominently [14,8,12]. Valiant's new theory manages to create a process of custom made cancellation which gives polynomial time algorithms for combinatorial problems which do not appear to be linear algebraic.

In terms of its broader impact in complexity theory, one can view Valiant's new theory as another algorithmic design paradigm which pushes back the frontier of what is solvable in polynomial time. Admittedly, at this early stage, it is still premature to say what drastic consequence it might have on the landscape of the big questions of complexity theory, such as P vs. NP. But the new theory has already been used by Valiant to devise polynomial time algorithms for a number of problems for which no polynomial time algorithms were known before.

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Unless and until a proof of P $\neq$ NP is found, one should regard this as an open problem. We can ask ourselves on what basis we derive confidence on the truth of this conjecture. In our view this confidence is not based on any partial lower bounds which are either for very restricted models of computation or are still very weak. Fundamentally this source of confidence in P $\neq$ NP comes from the fact that all existing algorithmic approaches do not seem to tackle a myriad of NP-complete problems. Valiant's new theory of holographic algorithms challenges us to re-examine this belief critically.

The theory is quite unlike anything before, and it is a delicate theory that will be difficult to explain without all the definitions. The central theorem in this theory is the beautiful *Holant Theorem*, which is the linchpin that holds everything together and makes it all possible. But, at least to us, the actual proof of the theorem in [17] was a little mysterious and somewhat difficult to understand. We believe the source of this difficulty lies in the way how one defines the main concepts of the theory.

The main purpose of this paper is to give a development of the theory based on the concept of *tensors*. While *tensor product* as an *operation* was already used by Valiant in [17], here our viewpoint is different in that we start off with the concepts of *covariant* and *contravariant* tensors, and, as it is customary in modern geometry, we strive to give it a *coordinate free* framework. Then various transformations of these tensors follow from general principles in tensor space. We then give a tensor theoretic proof of Valiant's Holant Theorem. It is suggested that once we have properly defined all the concepts based on *covariant* and *contravariant* tensors, Valiant's beautiful Holant Theorem can be understood as a *natural* expression of tensors.

Given the conceptual clarity afforded by the tensor perspective, we can easily see some generalizations of the *Holant Theorem* which follow from this framework.

2. Valiant's definitions

In this section we give a brief account of the key definitions of Valiant's theory, starting with the matching problem. More details can be found in [17].

Given a graph G, a matching of G is a set of edges no two of which share a vertex. A perfect matching M is a matching such that every vertex of G is incident to one edge of M. The decision problem of whether there is a perfect matching in G is computable in P, one of the notable achievements in the study of Algorithms. However, it is known that counting the number of perfect matchings in G is #P-complete.

We assign to every edge $e = (i, j)$ a variable x_{ij}, where $i < j$. Then we define the following polynomial

$$\mathrm{PerfMatch}(G) = \sum_{M} \prod_{(i,j)\in M} x_{ij},$$

where the sum is over all perfect matchings M. PerfMatch(G) is a polynomial on $\binom{n}{2}$ many variables x_{ij}, $1 \le i < j \le n$. If the graph is a weighted graph with weights w_{ij}, we can also evaluate PerfMatch(G) at $x_{ij} = w_{ij}$. Note that if all the weights are 1, then PerfMatch(G) just counts the number of perfect matchings in the graph.

A most remarkable result due to Fisher, Kasteleyn and Temperley (FKT) ([13,9], and [10]), from statistical physics is that for planar graphs, this Perfect Matching polynomial PerfMatch(G) can be evaluated in polynomial time. In fact it can be evaluated as a Pfaffian of a skew-symmetric matrix which is constructible from a planar embedding of G in polynomial time.

In effect, Valiant's theory allows the expression of a desired computation as an exponential sum, called the *Holant*, and via the Holant Theorem, reduces to the problem of computing the number of perfect matchings on planar graphs. This is done via the evaluation of PerfMatch(G) by the FKT method, for a suitably constructed *Matchgrid*, composed of *matchgates*, which we proceed to define. These reductions are called holographic reductions, because they carry out exponentially many cancellations analogous to a pattern of interference in quantum computing.

Define a *planar matchgate* Γ as a triple (G, X, Y) where G is a planar embedding of a weighted planar graph (V, E, W), $X \subseteq V$ is a set of input nodes, $Y \subseteq V$ is a set of output nodes, and $X \cap Y = \emptyset$. Furthermore in the planar embedding of G, counter-clockwise one encounters vertices of X, labeled $1, \ldots, |X|$ and then vertices of Y, labeled $|Y|, \ldots, 1$.

Valiant defines the standard signature, $u(\Gamma)$, of Γ to be a $2^{|X|} \times 2^{|Y|}$ matrix whose entries are indexed by subsets $X' \subseteq X$ and $Y' \subseteq Y$, and the entry indexed by (X', Y') is PerfMatch($G - Z$), where $Z = X' \cup Y'$. Here $G - Z$ denotes the subgraph of G obtained by removing the subset of nodes in Z (and all their incident edges). We will make