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Generating labeled planar graphs uniformly at random

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Abstract

We present a deterministic polynomial time algorithm to sample a labeled planar graph uniformly at random. Our approach uses recursive formulae for the exact number of labeled planar graphs with n vertices and m edges, based on a decomposition into 1-, 2-, and 3-connected components. We can then use known sampling algorithms and counting formulae for 3-connected planar graphs.

Keywords: Labeled planar graph; Enumeration; Decomposition; Sampling algorithm; Dynamic programming; Graph theory

1. Introduction

A *planar graph* is a graph which can be embedded in the plane, as opposed to a *map*, which is an embedded graph. There is a rich literature on the enumerative combinatorics of maps, starting with Tutte's census papers, e.g. [27]. An efficient random sampling algorithm was developed by Schaeffer [24]. Little is known about random planar graphs, although they recently attracted much attention [2,4,6,8,14,19,22]. If we had an efficient algorithm to sample a planar graph uniformly at random, we could experimentally verify conjectures about properties of random planar graphs. We could also use it to evaluate the average-case running times of algorithms on planar graphs. Denise, Vasconcellos, and Welsh [8] introduced a Markov chain whose stationary distribution is the uniform distribution on all labeled planar graphs. However, its mixing time is unknown and seems hard to analyze, and is perhaps not polynomial. Moreover, the corresponding sampling algorithm only approximates the uniform distribution.

We obtain the first deterministic polynomial time algorithm for generating a labeled planar graph uniformly at random.

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Theorem 1. *Labeled planar graphs on n vertices and m edges can be sampled uniformly at random in deterministic time $\tilde{O}(n^7)$ and space $O(n^4 \log n)$. If we apply a preprocessing step, this can also be done in deterministic time $\tilde{O}(n^3)$ and space $O(n^7)$.*

Our result uses known graph decomposition and counting techniques [28,31] to reduce the counting and random sampling of labeled planar graphs to the counting and random sampling of 3-connected *rooted planar maps*. Usually, a planar graph has many embeddings that are non-isomorphic like maps, but some graphs have a unique embedding. A classical theorem of Whitney (see e.g. [10]) asserts that 3-connected planar graphs are *rigid* in the sense that all embeddings in the sphere are combinatorially equivalent. As *rooting* destroys any further symmetries; they are closely related to 3-connected *labeled* planar graphs. Moreover, the 'degrees of freedom' of the embedding of a planar graph are governed by its connectivity structure. We exploit this fact by composing a planar graph out of 1-, 2-, and 3-connected components.

Our sampling procedure first determines the number of components, and how many vertices and edges they shall contain. Each connected component is generated independently from the others, but has the chosen numbers of vertices and edges. To generate a connected component with given numbers of vertices and edges, we decide on a decomposition into 2-connected subgraphs, and how the vertices and edges shall be distributed among its parts. So far, this approach is similar to the one used in [5], where the goal was to sample random outerplanar graphs. In the planar case, we need to go one step further.

Trakhtenbrot [26] showed that every 2-connected graph is uniquely composed of special graphs (called *networks*) of three kinds. Such networks can be combined in series, in parallel, or using a 3-connected graph as a core (see Theorem 2 below). Using this composition, we can then employ known results about counting and the random sampling of 3-connected planar maps.

The concept of rooting plays an important role in the enumeration of planar maps. A *face-rooted* map is one with a distinguished edge which lies on the outer face, to which a direction is assigned. Rooting forces isomorphisms to: map the outer face to the outer face, keep the root edge incident to the outer face, and preserve its direction. The enumeration of 3-connected face-rooted unlabeled maps with given numbers of vertices and faces – also called *c-nets* – was achieved by Mullin and Schellenberg [20]. We invoke their closed formulae in order to count 3-connected labeled planar graphs with given numbers of vertices and edges. For the generation of 3-connected labeled planar graphs with given numbers of vertices and edges, we employ a recent deterministic polynomial time algorithm [3]. Alternatively, we can use a sampling procedure that runs in linear time that was recently presented in [13]; in this case we obtain an *expected* polynomial time sample for labeled planar graphs, which runs in time $O(n^3)$ and space $\tilde{O}(n^6)$ after a preprocessing step in time $\tilde{O}(n^6)$, or in time $\tilde{O}(n^6)$ and space $O(n^4 \log n)$ without preprocessing.

When we apply the various counting sampling subroutines along the stages of the connectivity decomposition, we must branch with the correct probabilities. To compute those probabilities,we use recurrence formulae that can be evaluated in polynomial time using dynamic programming. Then the decomposition can be translated immediately into a sampling procedure.

The paper is organized as follows. In the next section we give the graph theoretic background for the decomposition of planar graphs, which guides us when we derive counting formulae for planar graphs in the following three sections. In Section 7 we analyze the running time and memory requirements of the corresponding sampling procedures, and discuss results from an implementation of the counting part. We conclude with a discussion of variations of the approach.

2. Decomposition by connectivity

Let us recall and fix some terminology [10,28–30]. A *graph* will be assumed to be unoriented and *simple*, i.e., having no loops or multiple (also called *parallel*) edges; if multiple edges are allowed, the term *multigraph* will be used. We consider labeled graphs whose vertex sets are initial segments of $\mathbb{N}_0$.

Every connected graph can be decomposed into *blocks* by being split at cutvertices. Here a block is a maximal subgraph that is either 2-connected, or a pair of adjacent vertices, or an isolated vertex. The *block structure* of a graph G is a tree whose vertices are the cutvertices of G and the blocks of G (considered as vertices), where adjacency is defined by containment. Conversely, we will *compose* connected graphs by identifying the vertex 0 of one part with an arbitrary vertex of the other. A formal definition of composition operations is given at the end of this section.