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The intersection of algebra and coalgebra ☆

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Abstract

Presheaf categories are well-known to be varieties of algebras and covarieties of coalgebras. We prove the converse: if a category is a variety as well as a covariety, then it is a presheaf category. Our main result is that all coalgebras on a set functor H form a presheaf category iff H is a reduction of a polynomial functor.

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Keywords: Variety; Covariety; Presheaf category

1. Introduction

The aim of the paper is to prove the equation

$$\text{algebra} \cap \text{coalgebra} = \text{presheaves}$$

over many-sorted sets. In the more restrictive case of algebra and coalgebra over **Set** the equation is

$$\text{algebra} \cap \text{coalgebra} = \text{monoid actions}.$$

This shows that here, essentially, just sequential automata form the intersection of algebra and coalgebra. In fact, a sequential automaton can be viewed as an algebra, or as a coalgebra: the main ingredient, the next-state function

$$\delta : Q \times I \longrightarrow Q \quad (I = \text{the input set})$$

defines an algebra of the endofunctor $(-) \times I$ of **Set**, but by currying it

$$\hat{\delta} : Q \longrightarrow Q^I$$

one gets a coalgebra of the endofunctor $(-)^I$. Now suppose that M is a monoid, then the category of M-sets (i.e., sets with a monoid action of M) is a subcategory of the category of sequential automata with the input set $I = M^*$, and

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M-sets are easily seen to be both a variety and a covariety of sequential automata. Is this a unique such situation, or are there other interesting examples of coalgebras that are algebras? A surprisingly general example was discovered by James Worrell: he proved in [21] that for every (not necessarily finitary) signature Σ we can view Σ-coalgebras, i.e., coalgebras of the *polynomial endofunctor* of **Set**

$$H_\Sigma Q = \coprod_{\sigma\in\Sigma} Q^n \quad (n = \text{arity of } \sigma)$$

as a variety of algebras. In fact, the category **Coalg** H_Σ of all Σ-coalgebras is equivalent to a presheaf category $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ for some small category $\mathcal{A}$, see [21]. Now $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is always a variety of unary algebras—but not always one-sorted! Thus, the slogan

all Σ-coalgebras form a variety of algebras

is, in general, only true if we move from one-sorted sets to many-sorted ones. Therefore, in the present paper we consider algebra and coalgebra over many-sorted sets (given by endofunctors of $\mathbf{Set}^S$, the category of S-sorted sets). Our results also hold for base categories of the form $\mathbf{Set}^{\mathcal{C}}$ where $\mathcal{C}$ is a small category, see 5.3.

We are going to describe the intersection of algebra and coalgebra, i.e., those categories which are at the same time varieties of F-algebras and covarieties of G-coalgebras for endofunctors F and G of many-sorted sets. We consider all these categories as *concrete categories*, i.e., pairs consisting of a category $\mathcal{V}$ and a faithful ("forgetful") functor $V: \mathcal{V} \longrightarrow \mathbf{Set}^S$. Given two concrete categories $V_i: \mathcal{V}_i \longrightarrow \mathbf{Set}^S$ for $i = 1, 2$ we call them *concretely equivalent* if there exists an equivalence functor $E: \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ such that V_1 is naturally isomorphic to $V_2 \cdot E$; notation $\mathcal{V}_1 \simeq \mathcal{V}_2$ (see [15]).

We will strengthen the result of James Worrell in several directions:

(1) Considering presheaf categories $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ as concrete categories over **Set**, the category of Σ-coalgebras is *concretely* equivalent to a presheaf category. And we prove the converse, which is our main result: given an endofunctor H of **Set** such that **Coalg** H is concretely equivalent to a presheaf category, then H is a reduction of a polynomial functor—thus **Coalg** H is the category of Σ-coalgebras for some Σ. ("Reduction" means that the value at the empty set can be changed.)

(2) Considering $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ as a concrete category over $\mathbf{Set}^S$, where S is the set of objects of $\mathcal{A}$, we prove that the presheaf category is *always* concretely equivalent to a covariety of coalgebras. And conversely: every many-sorted variety concretely equivalent to a many-sorted covariety is a category of presheaves.

(3) In contrast to (2), only very special small categories $\mathcal{A}$ have the property that $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is concretely equivalent to **Coalg** H over **Set**: $\mathcal{A}$ has to be equivalent to the Σ-tree category for some signature Σ. This category has all Σ-trees (see Example 2.7) as objects, and morphisms from t' to t are all nodes of t whose subtree (in t) is t'.

For all these results we work with concrete categories over $\mathbf{Set}^S$. The fact that Worrell's result about polynomial endofunctors of **Set** can be strengthened as in (1) above makes heavy use of concrete equivalence: we do not know the answer to the:

Open problem. For which endofunctors H of $\mathbf{Set}^S$ is the category **Coalg** H equivalent to a presheaf category?

The present paper is an expanded version of the paper [1] presented at the conference "Algebra and Coalgebra in Computer Science", CALCO 2005, in Swansea.

2. Varieties and covarieties

Remark 2.1. What is a many-sorted variety? Each of the following is a reasonable answer, depending on the generality one has in mind:

(a) An equationally presentable category of Σ-algebras, where Σ is a finitary, S-sorted signature, see [6] for the one-sorted case, and [7,13] for the many-sorted case.

(b) As above, but dropping "finitary". Thus, an *S-sorted signature* is a set Σ together with an *arity* of every operation symbol $\sigma \in \Sigma$ of the form

$$\sigma: (s_i)_{i<n} \longrightarrow s,$$