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Missing sets in rational parametrizations of surfaces of revolution

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HIGHLIGHTS

- We give a simple description of the missing area of ruled surface parametrization.
- We provide algorithms to compute, in each case, the missing area.
- We analyze the real and the complex case.

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ABSTRACT

Parametric representations do not cover, in general, the whole geometric object that they parametrize. This can be a problem in practical applications. In this paper we analyze the question for surfaces of revolution generated by real rational profile curves, and we describe a simple small superset of the real zone of the surface not covered by the parametrization. This superset consists, in the worst case, of the union of a circle and the mirror curve of the profile curve.

1. Introduction

Parametric representations of structured surfaces like ruled surfaces, surfaces of revolution or swept surfaces are often used in computer graphics, CAD/CAM, and surface/geometric modeling (see e.g. [1,2]). Nevertheless, when working with parametric instead of implicit representations, one must take into account that some information of the geometric object can be missed. More precisely, the parametrization may not cover the whole object, that is, some part of the object may not be reachable by giving values to the parameters; for instance, the curve parametrization $(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1})$ covers the unit circle with the exception of the point $(0, 1)$. In general, a rational parametrization, of a variety V, may not cover all V. The missing part is a constructible set of V (see [3, Theorem 3.16, p. 39]); that is, roughly speaking, the missing set is included in a finite union of proper closed subsets of V. For the particular case of curves, a parametrization may miss, indeed, at most, one point, called the critical point (see [4] or [5]). However, a surface parametrization may miss finitely many curves and finitely many points. We will refer to the uncovered part as the *missing set* of the parametrization.

We observe that the phenomenon described above can be seen as a particular case of the geometric covering problem (see e.g. [6]), in the sense that the image of the parametrization is the subset that one guard covers, and the missing set is the inspection location to be covered by other guards.

Parametrizations with nonempty missing sets can be a problem in practical applications if there is relevant information outside the covered part. Examples of this claim can be found in [7] (for the computation of intersections), in [8] (for estimating Hausdorff distances) or [9] (for the analysis of cross sections). One way to deal with this difficulty is to find parametrizations that do cover the whole object. In the curve case, there are algorithmic methods for that (see [5]). However, the situation for surfaces is much more complicated, and, at least to our knowledge, it is an open problem. Instead, one may use other alternatives. For instance, in [10,9,8], the authors compute finitely many parametrizations such that their images cover all the surface. Another possibility is to have a precise description of the missing set of the parametrization, or a subset of the surface containing the missing set; a subset of the surface, containing the missing set and having dimension smaller than 2, is called a *critical set*. In this way, for a practical application

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one can use the parametrization and then decide the existence of relevant points in the critical set.

The last strategy can be approached by using elimination theory techniques (see [7]). Nevertheless, although theoretically possible, the direct use of these techniques produces, in general, huge critical sets and requires solving systems of algebraic equations. As a consequence, the method turns to be inefficient in practice. However, when working with structured surfaces, a preliminary analysis of the structure can help to describe quickly and easily a critical set. For instance, in [8], we show that any rational ruled surface can be parametrized so that the critical set is a line which is easily computable from the parametrization. In this paper, we analyze the case of surfaces of revolution given by means of a real plane curve parametrization known as a profile. We prove that a critical set for the real part of a surface of revolution is, in the worst case, the union of a curve (the mirror curve of the profile curve) and a circle passing through the critical point of the profile curve; see Table 1. As a direct criterion (see Corollary 2.2), we obtain that any parametrization of a symmetric real curve with at least one polynomial component generates all the real part of the surface of revolution.

As we will see in the subsequent sections this critical set is indeed very simple to compute from the profile curve parametrization. An additional advantage of our method is that it does not require that the parametrization of the surface is proper (i.e. injective), while the direct application of elimination techniques needs to compute the inverse of the parametrization, and hence requires that the surface parametrization is proper.

The rest of the paper is organized as follows. In Section 2 we present the main results of the paper. The proofs of these results appear in the appendix. In Section 3 we outline the algorithmic methods derived from the theoretical results, and we illustrate them by some examples. Future work on the topic is discussed in Section 4. The paper ends with a brief conclusion.

Computations were performed with the mathematical software Maple 18. Plots were generated with Maple and Surfer.

2. Results

Let $\mathbf{C}^{\mathrm{P}}$ be a curve (*profile curve*) in the (y, z)-plane parametrized by $\mathbf{r}^{\mathrm{P}}(t) = (0, p(t), q(t))$, where $p(t), q(t)$ are rational functions with real coefficients; the results presented here are also valid if the coefficients are complex numbers but for simplicity, and because of the interest in applications, we limit the setting to the real case. In addition, we assume that $\mathbf{r}^{\mathrm{P}}$ is proper, that is, injective. We observe that every non-proper parametrization can be reparametrized into a proper one (see for example Section 6.1. in [11]). Also let $\mathbf{S}$ be the surface of revolution generated by rotating $\mathbf{C}^{\mathrm{P}}$ around the z-axis. We exclude the trivial case where $\mathbf{C}^{\mathrm{P}}$ is a line parallel to the y axis, in which $\mathbf{S}$ is a plane. The classical parametrization of $\mathbf{S}$, obtained from $\mathbf{r}^{\mathrm{P}}(t)$, is

$$\mathbf{P}(s,t) = \left(\frac{2s}{1+s^2}\,p(t),\ \frac{1-s^2}{1+s^2}\,p(t),\ q(t)\right).$$

Observe that properness is assumed in the profile parametrization $\mathbf{r}^{\mathrm{P}}(t)$ but not in $\mathbf{P}(s, t)$; see Example 2.3 in [12] for an example where $\mathbf{P}(s, t)$ is non-proper and $\mathbf{r}^{\mathrm{P}}(t)$ is proper.

In addition, we consider the parametric curve $\mathbf{C}^{\mathrm{M}}$ (called *mirror curve of* $\mathbf{C}^{\mathrm{P}}$) parametrized as $\mathbf{r}^{\mathrm{M}}(t) = (0, -p(t), q(t))$. Observe that $\mathbf{C}^{\mathrm{P}} = \mathbf{C}^{\mathrm{M}}$ if and only if $\mathbf{C}^{\mathrm{P}}$ is symmetric with respect to the z-axis. For instance, the parabola $y = z^2$ is equal to its mirror curve while the cubic $y = z^3$ is not. Finally, we represent by $\mathbf{circ}(\alpha, c)$ the circle of radius $|\alpha|$ in the plane $z = c$ centered at $(0, 0, c)$, that is, the curve parametrized as

$$\left(\frac{2s}{1+s^2}\,\alpha,\ \frac{1-s^2}{1+s^2}\,\alpha,\ c\right).$$

Observe that $\mathbf{P}(s, t_0)$ is $\mathbf{circ}(p(t_0), q(t_0))$, i.e. the *cross section circle* of the surface $\mathbf{S}$ of revolution passing through $(0, p(t_0), q(t_0))$.

Before, stating our main results, we need to recall the notion of normal (i.e. surjective) curve parametrization and critical point, for further details see [5,4]. We say that a curve parametrization $\mathbf{r}(t)$ is normal if all points on the curve are reachable by $\mathbf{r}(t)$ when t takes values in the field of the complex numbers. The theory establishes that a proper curve parametrization can miss at most one point. This point is called the *critical point*, and it can be seen, in the case of real functions, as the limit, when t goes to ∞, of the parametrization; understanding that if this limit does not exist then there is no critical point and the parametrization is normal. For instance, (t, t^2) or $(t, 1/t)$ are normal, but the circle parametrization

$$\left(\frac{2t}{t^2+1},\ \frac{t^2-1}{t^2+1}\right)$$

has the *East pole* $(0, 1)$ as critical point; indeed, it is not reachable and hence the parametrization is not normal. In this situation, we are ready to establish our main results (see Appendix for the formal proof of the results).

2.1. The real case

We describe now a critical set of the real part of $\mathbf{S}$ that $\mathbf{P}(s, t)$ does not cover. For this purpose, we distinguish whether the profile curve is symmetric or not. The next theorem states that, in the symmetric case, at most one point can be missed in the real part of the surface of revolution.

Theorem 2.1 (*Symmetric Real Case*)**.** *Let $\mathbf{C}^{\mathrm{P}}$ be symmetric.*

1. *If $\mathbf{r}^{\mathrm{P}}(t)$ is normal, the empty set is a real-critical set of $\mathbf{P}(s, t)$.*
2. *If $\mathbf{r}^{\mathrm{P}}(t)$ is not normal, and $(0, b, c)$ is its critical point, then $\{(0, b, c)\}$ is a real-critical set of $\mathbf{P}(s, t)$.*

Based on the previous theorem and on Theorem 2.8 in [13] one has the following corollary.

Corollary 2.2. *If $\mathbf{r}^{\mathrm{P}}$ is symmetric, and at least one of its components has a numerator of degree greater than the degree of the denominator, then $\mathbf{P}(s, t)$ covers all $\mathbf{S}$.*

The next theorem states that, in the non-symmetric case, the missing real part of the surface of revolution is included in the union of the mirror curve and either the critical point of the profile curve or a cross-section circle.

Theorem 2.3 (*Non-Symmetric Real Case*)**.** *Let $\mathbf{C}^{\mathrm{P}}$ be non-symmetric.*

1. *If $\mathbf{r}^{\mathrm{P}}(t)$ is normal, $\mathbf{C}^{\mathrm{M}}$ is a real-critical set of $\mathbf{P}(s, t)$.*
2. *If $\mathbf{r}^{\mathrm{P}}(t)$ is not normal, and $(0, b, c)$ is its critical point, then a real-critical set of $\mathbf{P}(s, t)$ is $\mathbf{C}^{\mathrm{M}}$ if $(0, -b, c) \in \mathbf{C}^{\mathrm{P}}$, otherwise $\mathbf{C}^{\mathrm{M}} \cup \mathbf{circ}(b, c)$ is.*

2.2. The complex case

Next, we describe a critical set when the revolution surface in embedded in the complex space. The next theorem states that, in the (complex) case, besides the real missing part introduced in Theorems 2.1 and 2.3, one may miss pairs of complex lines settled at each (real or complex) intersection of $\mathbf{C}^{\mathrm{P}}$ with the z-axis.

Theorem 2.4 (*Complex Case*)**.** *Let $\mathcal{A}$ be the real critical set of $\mathbf{P}(s, t)$ provided by Theorems 2.1 and 2.3, and let J be the set of all (real and complex) z-coordinates of the intersection points of $\mathbf{C}^{\mathrm{P}}$ with the z-axis. Then, a complex-critical set of $\mathbf{P}(s, t)$ is*

$$\mathcal{A} \bigcup_{\lambda \in J} \{(t, \pm i t, \lambda) \mid t \in \mathbb{C}\}.$$