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Tolerance analysis by polytopes: Taking into account degrees of freedom with cap half-spaces[☆]

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HIGHLIGHTS

- In tolerance analysis, sets of constraints can be compliant with operand polyhedra.
- These operands are generally unbounded due to the inclusion of degrees of freedom.
- Cap half-spaces are added to each polyhedron to make it compliant with a polytope.
- The influence of the cap half-spaces on the topology of polytopes must be controlled.
- It is necessary to ensure the compliance of a mechanism in terms of requirements.

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ABSTRACT

To determine the relative position of any two surfaces in a system, one approach is to use operations (Minkowski sum and intersection) on sets of constraints. These constraints are made compliant with half-spaces of $\mathbb{R}^n$ where each set of half-spaces defines an operand polyhedron. These operands are generally unbounded due to the inclusion of degrees of invariance for surfaces and degrees of freedom for joints defining theoretically unlimited displacements. To solve operations on operands, Minkowski sums in particular, "cap" half-spaces are added to each polyhedron to make it compliant with a polytope which is by definition a bounded polyhedron. The difficulty of this method lies in controlling the influence of these additional half-spaces on the topology of polytopes calculated by sum or intersection. This is necessary to validate the geometric tolerances that ensure the compliance of a mechanical system in terms of functional requirements.

1. Introduction

A first approach to handling geometric tolerancing is generally a parametric approach [1]. Parametric approaches, especially those used in the various commercial tools, formalize the relative position of any two surfaces of a mechanism at a specific point by a simple relation (linear or non-linear) between parameters of position (translation and/or rotation). This relation is obtained using either an analytical method [2–5] or a Monte Carlo method [6]. Several works on this subject are based on TTRS and MGDE concepts [7]. The GapSpace approach has been proposed to ensure the assembly conditions [8].

This type of approach does not take into consideration the dependencies between the limits of rotation parameter and translation parameter. Nor does it support the redundancy of suppressing degrees of freedom (hyperstatism) between two parts.

A solution to address these limitations is to introduce sets of constraints [9–15]. These sets of constraints define the boundaries of relative displacements between two surfaces of the same part (geometric constraints) and boundaries of relative displacements between two surfaces of two separate parts, but which are potentially in contact (contact constraints). In general the sets of constraints manipulated by clearance space are not linear [11]. This article focus on the approach by constraints and more precisely on bounded sets of constraints made up of linear constraints [12].

Sets of geometric and contact constraints are generally operand sets which can be made compliant with finite sets of half-spaces of $\mathbb{R}^n$ [12]. The boundaries of relative displacements between two surfaces result from the intersection of the half-spaces of $\mathbb{R}^n$ of an operand set, defining an operand polyhedron. A polyhedron is not usually bounded, due to the degree of invariance of a surface or

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Symbols and Abbreviations

H	hyperplane of $\mathbb{R}^6$
$\overline{H}^-$	negative closed half-space associated to H in $\mathbb{R}^6$
$\overline{H}^+$	positive closed half-space associated to H in $\mathbb{R}^6$
$\overline{Hc_k}^-$	k^{th} cap half-space in $\mathbb{R}^6$; $\overline{H_k}^-$ the k^{th} non-cap half-space in $\mathbb{R}^6$
n_{nc}	number of non-cap half-spaces in a polytope of $\mathbb{R}^6$
n_c	number of cap half-spaces in a polytope of $\mathbb{R}^6$
$\overline{H_{i,k}}^-$	k^{th} non-cap half-space of polytope i.
$\overline{Hc_{i,k}}^-$	k^{th} cap half-space of polytope i.
$N(\mathcal{P}_i)$	normal fan of polytope $\mathcal{P}_i$
$N^R(\mathcal{P}_i)$	normal fan of the polyhedron containing $\mathcal{P}_i$.
$\text{DualCone}\left(v_{ij}\right)$	dual cone associated to vertex v_{ij}
v_{ij}	j^{th} vertex of polytope $\mathcal{P}_i$.
$\mathcal{P}^f$	functional polyhedron
$t_{i,j}$	dimension of the tolerance zone containing the surface i, j
$t_{\text{N-}x}$	translation at point N along x-axis
r_x	rotation along x-axis
$\mathbf{N}_i\mathbf{M} \times \mathbf{r}$	cross product of vectors $\mathbf{N}_i\mathbf{M}$ and $\mathbf{r}$

to the degree of freedom of joints defining theoretically unlimited displacements [16,17].

The relative position between any two surfaces of a mechanism is determined by operations on these operand polyhedrons (Minkowski sum and intersection), [17–19]. To solve these operations, and in particular the Minkowski sums in $\mathbb{R}^n$, one method is to delimit the intersection of the half-spaces of an operand set and thus transform a polyhedron of $\mathbb{R}^n$ into a polytope of $\mathbb{R}^n$, as a polytope of $\mathbb{R}^n$ is a bounded polyhedron of $\mathbb{R}^n$ [20]. This method is justified by the algorithmic complexity of summing polyhedra of $\mathbb{R}^n$, requiring the development of algorithms to compute tolerance analysis of the Minkowski sums of polytopes [21–23].

The main contribution of this article is to introduce the concept of "cap" half-spaces to bound operand polyhedra. They are added to the operand set and in this way determining the relative position of two surfaces of a mechanical systems is based solely on operations on operand polytopes generating a calculated polytope [24].

By checking that a calculated polytope is included within a functional polytope the conformity of a mechanical system can be simulated with respect to a functional requirement [12], see Fig. 1.

The addition of cap half-spaces to the operand sets will affect the topology of a calculated polytope. Hence it has to be possible to differentiate among all the facets of a calculated polytope between those that are generated by the cap half-spaces and the others generated by half-spaces that derive from geometric and contact constraints. This is essential in order to validate the geometric tolerances that ensure that a mechanical system is compliant in relation to a functional requirement.

This article describes how to identify the facets generated by the cap half-spaces of a polytope resulting from a Minkowski sum or an intersection between two operand polytopes.

In the following, we limit ourselves to 6-dimension polyhedra and polytopes: the half-spaces arising from the geometric and contact constraints are linear inequalities in six variables: three rotation variables and three translation variables [24].

In the first part, some properties of polyhedra and polytopes are considered; the second part looks at determining the cap half-spaces which set boundaries to the half-space intersections resulting from geometric and contact constraints.

The third part deals with the two methods of identifying the dependent facets of cap half-spaces in a summation and an intersection respectively. An example of an application of tolerance analysis illustrating these two methods is described at the end of the article.

For the application used as an example, we put forward the following physical hypotheses:

- no form defect in the real surfaces,
- no local strain in surfaces in contact,
- no deformable parts.

2. Some definitions and properties of polyhedra and polytopes

We first set out some definitions and properties of polyhedra and polytopes to ensure a proper understanding of the rest of the article. These are taken from [20].

2.1. Hyperplane, half-space

A hyperplane is an affine subspace of dimension 0, 1, 2 or $(n-1)$ in $\mathbb{R}^n$ called point, line, plane and hyperplane respectively. A hyperplane of dimension $(n-1)$ is denoted a $(n-1)$-hyperplane.

Let us consider H a hyperplane in some $\mathbb{R}^n$: $a_1x_1 + \cdots + a_ix_i + \cdots + a_nx_n = b$.

$$H = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\right\} \quad \text{with } \mathbf{a}^T = (a_1, \ldots, a_n) \in \mathbb{R}^n,$$
$$\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \text{ and } b \in \mathbb{R}. \tag{1}$$

For each hyperplane H we define the positive closed half-space $\overline{H}^+$ by:

$$\overline{H}^+ = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = a_1x_1 + \cdots + a_ix_i + \cdots + a_nx_n \geq b\right\} \tag{2}$$

and the negative closed one $\overline{H}^-$ similarly:

$$\overline{H}^- = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = a_1x_1 + \cdots + a_ix_i + \cdots + a_nx_n \leq b\right\} \tag{3}$$

In the following any manipulated constraint will be written in terms of $\overline{H}^-$ denoted simply:

$$\overline{H}^- = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leq b\right\}. \tag{4}$$

2.2. Polyhedron, polytope

A polyhedron $\mathcal{P}$ is the intersection of a finite number of closed half-spaces of $\mathbb{R}^n$. We define:

$$\mathcal{P} = \bigcap_{i=1}^{m} \overline{H_i}^- = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a_i}^T\mathbf{x} \leq b_i \; i = 1, \ldots, m\right\}$$
$$= \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\right\}, \quad \mathbf{A} \in \mathbb{R}^{m\times n} \tag{5}$$

where $\mathbf{a_i}^T$ is the i^{th} line of $\mathbf{A}$ and b_i the i^{th} component of $\mathbf{b}$.

This definition characterizes the $\mathcal{H}-$ description of $\mathcal{P}$, see Fig. 2(a), [20,25] and [26].

A hyperplane H is said to be a support hyperplane for polyhedron $\mathcal{P}$ if and only if:

$$\mathcal{P} \cap H \neq \emptyset \quad \text{and} \quad \mathcal{P} \subset \overline{H}^-. \tag{6}$$

A face F of $\mathcal{P}$ is the intersection of the polyhedron $\mathcal{P}$ with one of its support hyperplanes. The faces of a d-polyhedron $\mathcal{P}$ are convex sub-sets of dimension k, $0 \leq k \leq d-1$.

A face of dimension k is called k-face. A 0-face is a vertex, a 1-face is an edge and a $(d-1)$-face is a facet of $\mathcal{P}$.