Computer-Aided Design 62 (2015) 218–226

Contents lists available at ScienceDirect

Computer-Aided Design

journal homepage: www.elsevier.com/locate/cad

A polynomial Hermite interpolant for C^2 quasi arc-length approximation☆

J. Sánchez-Reyes*, J.M. Chacón

University of Castilla-La Mancha, IMACI, ETS Ingenieros Industriales, Spain

HIGHLIGHTS

- Transcendental curves and most offsets do not admit exact NURBS representation.
- We apply Hermite interpolation to achieve C^2 quasi arc-length approximation.
- Two alternative tools are considered: piecewise Bézier quintics and cubic B-splines.
- The quintic displays simple control points, with locally nonparametric arrangement.
- We approximate offsets and clothoids and compare our results with existing software.

ARTICLE INFO

Article history:
Received 6 May 2014
Accepted 2 December 2014

Keywords:
Arc-length
Clothoid
Hermite interpolation
Offset
Quintic
Transcendental curve

ABSTRACT

Transcendental curves, or in general those resulting from offsetting, do not admit an exact rational or polynomial representation and must hence be approximated in order to incorporate them into most commercial CAD systems. We present a simple, yet general geometric tool for polynomial approximation, based on piecewise Hermite interpolation with C^2 quasi arc-length parameterization, a desirable property for robotics or CNC. We take the osculatory Hermite interpolation, prescribing position, tangent direction and curvature at the endpoints, and add quasi arc-length conditions, by imposing unit speed and vanishing tangential acceleration. These new conditions fit naturally into this scheme, yielding a quintic with Bézier points that turn out to display extremely simple geometry. In addition we consider a lower degree alternative to the quintic, namely a cubic B-spline. Finally, we include two examples of applications (the approximations of regular offsets and the clothoid) and compare our results with those from commercial systems or existing methods.

1. Introduction

Curves in CAD are usually represented by employing a polynomial or rational parameterization, expressed in Bernstein basis and its B-spline extension. However, remarkable cases, such as the offset or transcendental curves, do not admit this representation. In consequence, they cannot be incorporated directly into most commercial systems and must be approximated in some way. Another expressive shortcoming of the standard rational model [1–3] is that it cannot yield curves with exact arc-length parameterization [4,5], aside from the trivial case of a straight line.

Most approximation methods concentrate on generating a good approximant in the sense of being close to a given curve, by guaranteeing a maximum distance. However, this requisite does not suffice for applications requiring a smooth parameterization, where *smooth* usually means at least C^2 and close to the ideal arc-length (also called *intrinsic* or *natural* parameterization). An example where parameterization plays a key role is a trajectory whose parameter is taken as proportional to time. A smooth parameterization with approximately unit speed is desirable for CNC machining, or the definition of trajectories for robotic manipulators. Parameterization is also of paramount importance in *skinning* (traditionally known as *lofting*) [1], where several section curves are connected to generate a smooth surface. If the section curves are unevenly parameterized, the resulting surfaces display poor quality. Finally, even in case a curve admits an exact rational representation, such as the circle, a polynomial approximation may be required to improve its parameterization.

To achieve a quasi arc-length approximation to a given curve, a first option would be to sample points on the curve and construct an interpolant that tries to minimize its speed deviation from unity by some optimization technique, as done in [6,7] using a

☆ This paper has been recommended for acceptance by Dr. Vadim Shapiro.

* Corresponding author. Tel.: +34 926 295463; fax: +34 926 295361.
E-mail address: Javier.SanchezReyes@uclm.es (J. Sánchez-Reyes).

http://dx.doi.org/10.1016/j.cad.2014.12.001
0010-4485/

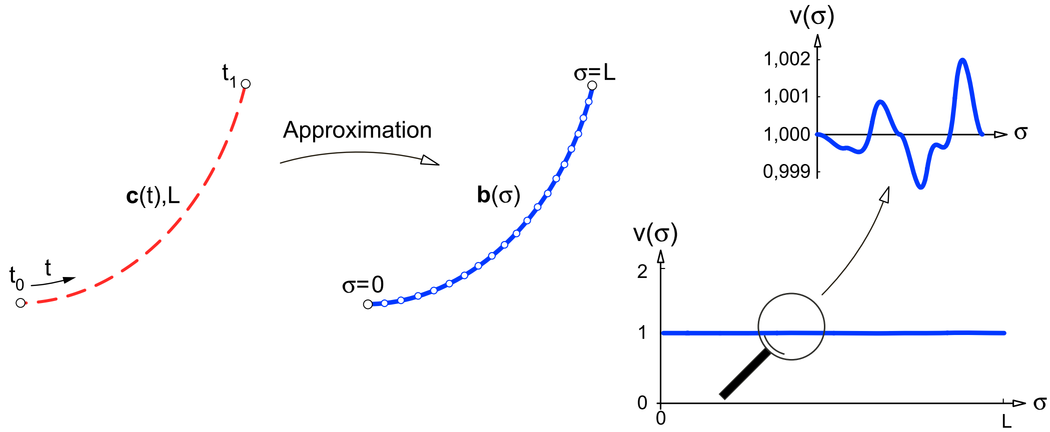

Fig. 1. Quasi arc-length approximation $\mathbf{b}(\sigma)$ to a given curve $\mathbf{c}(t)$.

quintic spline, or in [8] with a cubic B-spline. We advocate instead a simpler, more geometric alternative that reproduces not only positional data but also tangential and curvature information from the original curve, by using a modified Hermite interpolation technique. The remaining degrees of freedom of the interpolant are employed to impose the quasi arc-length condition, namely unit speed and vanishing tangential acceleration at the endpoints. Thus, our approach is aimed at representing in an approximate manner, with the bonus of a smooth parameterization, those curves that do not fit into the NURBS model.

The article is arranged as follows. First, in Section 2 we set the Hermite conditions for the interpolant and consider two alternative solutions (a piecewise Bézier quintic and a cubic B-spline). Remarkably, the quasi arc-length condition results in extremely simple geometry for the control points. We apply our tool to approximate regular offset curves (Section 3) and the clothoid (Section 4), a notable transcendental curve, comparing our results with those from commercial CAD software. Finally, conclusions and extensions of this work are outlined in Section 5.

2. Quasi arc-length approximation from Hermite conditions

2.1. Quasi arc-length Hermite conditions

The Hermite-like interpolation process we advocate is sketched in Fig. 1. Given an original curve segment $\mathbf{c}(t)$, with arbitrary parameterization over a general domain $t \in [t_0, t_1]$, we seek a quasi arc-length approximation $\mathbf{b}(\sigma)$, defined over an interval $\sigma \in [0, L]$ of length L equal to that of $\mathbf{c}(t)$, where σ approximates the arc-length parameter s. Therefore, the parametric speed $v(\sigma)$ should be close to the ideal unit speed:

$$v(\sigma) = |d\mathbf{b}/d\sigma| = |\dot{\mathbf{b}}(\sigma)| \approx 1, \quad \sigma \in [0, L].$$

Another characteristic of arc-length parameterization also admits a clear kinematic interpretation. For an arc-length curve $\mathbf{a}(s)$, since $\dot{\mathbf{a}}(s) \cdot \dot{\mathbf{a}}(s) = 1$, by differentiating this scalar product then $\dot{\mathbf{a}}(s) \cdot \ddot{\mathbf{a}}(s) = 0$. In other words, the curve is traversed with vanishing tangential acceleration.

The key idea is to add these kinematic requirements to the customary osculatory Hermite interpolation (Fig. 2(a)), by finding the Hermitian interpolant $\mathbf{b}(\sigma)$ that meets the following conditions at the endpoints $\{t_0, t_1\}$:

(i) Interpolation of positions, tangents directions $\mathbf{t}_0, \mathbf{t}_1$, and curvatures κ_0, κ_1 of $\mathbf{c}(t)$.
(ii) Unit speed v, and acceleration $\ddot{\mathbf{b}}$ with vanishing tangential component $\ddot{\mathbf{b}}^t$:

$$v(0) = v(L) = 1, \qquad \ddot{\mathbf{b}}^t(0) = \ddot{\mathbf{b}}^t(L) = 0. \tag{1}$$

Condition (i) ensures that $\mathbf{b}(\sigma)$ mimics the shape of $\mathbf{c}(t)$, whereas condition (ii) imposes a quasi arc-length parameterization. In total, 6 conditions are prescribed at each endpoint, so the interpolant must have a total of $6 \times 2 = 12$ degrees of freedom. Using a Bézier curve [3], this means 6 control points, i.e., a quintic.

If the approximation with a single interpolant is not satisfactory, we subdivide the initial segment $\mathbf{c}(t)$, $t \in [t_0, t_1]$, into several pieces and construct a new Hermite interpolant for each piece, as shown in Fig. 2(b), which furnishes a C^2 piecewise approximation. To generate a more compact C^2 quintic B-spline [3], rather than a piecewise Bézier representation, just knot together the Bézier segments, by following a two-step procedure:

1. Merge the approximations into a quintic B-spline curve, with de Boor points obtained by concatenating the Bézier polygons, and knot positions by concatenating each interval length, with knot multiplicity 5 equal to the degree.
2. Apply twice knot removal [1,9] at each internal knot, down to multiplicity 3.

Finally, Fig. 2 illustrates the case of a planar open segment, although it clearly extends to 3D curves, or closed curves by selecting a breakpoint corresponding to both t_0 and t_1.

2.2. Quintic Bézier interpolant

Next, we derive the Bézier points for the quintic interpolant, from the conditions (i) and (ii) explained in the previous section. Consider a quintic $\mathbf{b}(\sigma)$, with control points $\{\mathbf{b}_k\}_{k=0}^{5}$ and domain $\sigma \in [0, L]$, instead of the customary unit interval. If $\mathbf{t}_0$ denotes the unit tangent vector of the original curve $\mathbf{c}(t)$ at the endpoint t_0, then the interpolation of position and tangent, along with the requirement of unit speed (1) at t_0, yield directly the initial control points $\mathbf{b}_0, \mathbf{b}_1$:

$$\mathbf{b}_0 = \mathbf{c}(t_0), \qquad \mathbf{b}_1 = \mathbf{b}_0 + \frac{L}{5}\mathbf{t}_0. \tag{2}$$

Additionally in (ii), we impose a vanishing tangential component for the second derivative (1). This is tantamount to saying that $\mathbf{b}_0, \mathbf{b}_1$, along with the projection $\mathbf{b}_2^t$ of $\mathbf{b}_2$ onto the tangent line, define a Bézier line segment with vanishing second derivative at $\mathbf{b}_0$, and in consequence $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2^t$ lie equally spaced, as depicted in Fig. 3. Since the distance along the tangent line between $\mathbf{b}_0$ and $\mathbf{b}_1$ is $L/5$ (2), then $\mathbf{b}_1$ and $\mathbf{b}_2^t$ are also separated by $L/5$. Intuitively, the Bézier interpolant $\mathbf{b}(\sigma)$ is locally nonparametric [3], taking as abscissa axis the tangent line (and as ordinate axis the normal direction). Finally, the interpolation (i) of the curvature κ_0 of $\mathbf{c}(t)$ at $t = t_0$ yields the distance h from $\mathbf{b}_2$ to $\mathbf{b}_2^t$:

$$h = \frac{\kappa_0 L^2}{20}, \tag{3}$$