Computer-Aided Design 46 (2014) 45–57

Contents lists available at ScienceDirect

Computer-Aided Design

journal homepage: www.elsevier.com/locate/cad

Isogeometric analysis on triangulations

Noah Jaxon, Xiaoping Qian *

Department of Mechanical, Materials and Aerospace Engineering, Illinois Institute of Technology, Chicago, IL 60616, USA

HIGHLIGHTS

- Isogeometric analysis on triangulation of a domain bounded by NURBS curves.
- Geometry and solution represented by bivariate splines in Bernstein–Bézier form.
- Approach to construct parametric domain and construct C^r-smooth basis functions.
- Applicable to complex topologies and allow highly localized refinement.
- Isogeometric analysis of linear elasticity and advection–diffusion demonstrated.

ARTICLE INFO

Keywords:
Isogeometric analysis
Rational triangular Bézier
Bivariate splines
NURBS
Bernstein–Bézier form

ABSTRACT

We present a method for isogeometric analysis on the triangulation of a domain bounded by NURBS curves. In this method, both the geometry and the physical field are represented by bivariate splines in Bernstein–Bézier form over the triangulation. We describe a set of procedures to construct a parametric domain and its triangulation from a given physical domain, construct C^r-smooth basis functions over the domain, and establish a rational Triangular Bézier Spline (rTBS) based geometric mapping that C^r-smoothly maps the parametric domain to the physical domain and exactly recovers the NURBS boundaries at the domain boundary. As a result, this approach can achieve automated meshing of objects with complex topologies and allow highly localized refinement. Isogeometric analysis of problems from linear elasticity and advection–diffusion analysis is demonstrated.

1. Introduction

Isogeometric analysis is a technique of numerical analysis that uses basis functions commonly found in CAD geometries to represent both geometry and physical fields in the solution of problems governed by partial differential equations (PDE) [1,2]. Non-uniform rational *B*-splines (NURBS) are the *de facto* standard for geometric representation in CAD systems. The use of a NURBS-compatible basis in the solution of physical problems therefore leads to the elimination of geometric-approximation error in even the coarsest mesh. The increased continuity of the NURBS basis has led to significant numerical advantages over traditional Lagrange polynomials and other C^0 inter-element continuity based finite element analysis, e.g. improved convergence rate on a per degree-of-freedom (DOF) basis [2]. However, NURBS-based isogeometric analysis also faces challenges. For example, it is challenging to automatically construct NURBS-based volumetric representation of a complex physical domain since CAD geometries only contain boundary representation of the domain; Further, the tensor-product structure of NURBS makes it harder to perform local mesh refinement as is commonly desired during analysis.

Recently, significant progress has been made in addressing these challenges. For example, the swept volume [3], harmonic functions [4], multi-block [5], and Coons patch [6] techniques have recently been developed to construct NURBS representations of volumetric domains. To extend NURBS representation to complex topologies while also allowing for adaptive refinement, T-splines [7] have been used in isogeometric analysis [8–10]. Methods for constructing T-spline based parametrization of the domain are being developed [11,12]. Among alternate isogeometric representation and analysis techniques under development, a technique based on subdivision solids has recently been proposed [13]. Further, boundary-integral based isogeometric analysis techniques [14,15] seek to effectively bypass the need for volumetric parametrization.

We present an alternative approach to isogeometric analysis with the goal of achieving automatic discretization of the physical domain while eliminating geometric approximation error, allowing local refinement of the discretization and making it applicable to complex topologies. Our approach is based on triangulations of physical domains where both the geometry and physical field are represented by C^r-continuous multivariate splines in their Bernstein–Bézier form. In this paper, we restrict our attention to two-dimensional problems and bivariate splines. In our method, we first construct a polygonal parametric domain $\widehat{\Omega}$ that

* Corresponding author.
E-mail addresses: njaxon@hawk.iit.edu (N. Jaxon), qian@iit.edu (X. Qian).

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http://dx.doi.org/10.1016/j.cad.2013.08.017

mimics the NURBS-bounded physical domain Ω. We then obtain a triangulation T of $\widehat{\Omega}$, on which a Bernstein–Bézier form of a C^r bivariate spline basis is constructed. We use this basis to construct globally C^r-smooth geometric mapping that maps the parametric domain $\widehat{\Omega}$ to the physical domain Ω with exact recovery of the NURBS boundary. When exceptional vertices/edges are allowed, this approach also ensures global bijectivity of the mapping. We demonstrate our analysis results for linear elasticity and advection–diffusion problems on problems which are characteristically non-trivial to mesh by other methods. Since robust technologies for automatic triangulation with local refinement are currently available, our approach is fully automated, is applicable to objects of complex topologies and allows for local refinement during in the course of analysis.

Our work differs from prior work on multivariate-spline based analysis [16,17] in that we explicitly construct C^r-smooth bases and use rational Triangular Bézier Splines (rTBS) to ensure the exact recovery of the NURBS boundary. Our work also differs from the recent developed non-uniform rational Powell–Sabin splines for isogeometric analysis [18,19]. Our approach is more general since general C^r spline spaces are considered. Further, we use Bézier ordinates and the corresponding basis functions to represent PDE solutions. Therefore, our Béizer ordinates based representation has direct geometric interpretation. In contrast, the approach [18,19] uses Powell–Sabin triangles and the corresponding normalized Powell–Sabin B-splines to represent the solutions. However, Powell–Sabin triangles are not unique for a given triangulation although the normalized Powell–Sabin B-splines have nice computational properties such as negativity. Further, our approach is applicable to macroelements or non-macroelements alike and the approach in [18,19] is an macroelement based approach.

Fig. 1 gives a schematic overview of our proposed approach. A C^r continuous basis $\boldsymbol{\psi}(\boldsymbol{\xi})$ is constructed over the parametric domain $\widehat{\Omega}$. The basis is used to construct an rTBS based geometric map $\boldsymbol{G}(\boldsymbol{\xi})$ so that it maps a point $\boldsymbol{\xi} \in \mathbb{R}^2$ in parametric domain $\widehat{\Omega}$ to a point $\boldsymbol{x} \in \mathbb{R}^2$ in the physical domain Ω. The same basis is also used to approximate physical field $u(\boldsymbol{\xi})$. Composing the inverse of geometric map and the field approximation, $\boldsymbol{u} \circ \boldsymbol{G}^{-1}$, defines a field on the physical domain. Quadrature in analysis integration is performed via local barycentric coordinates on the parent triangle.

The remainder of this paper is organized as follows: Section 2 introduces necessary background concepts; Section 3 presents our discretization method—smooth rTBS-based discretization of the physical domain; Section 4 discusses the details of smooth rTBS-based isogeometric analysis; Section 5 contains our numerical results; In Section 6 we present our conclusions.

2. Background

In this section we briefly introduce the Bézier curve, non-uniform rational B-splines (NURBS) and triangular Béziers. We then discuss the splines over triangulations and the Clough–Tocher and Powell–Sabin splits. This introduction aims to make the paper self-contained and to clarify notation for subsequent sections. For further reading on Bézier curves, B-splines, and Bézier triangles, see [20], for splines on triangulations, see [21], and for isogeometric analysis, see [2].

2.1. Bézier and NURBS curves

CAD geometry is usually defined by a NURBS represented boundary. Each knot span of a NURBS curve corresponds to a Bézier curve. A Bézier curve is defined through Bernstein basis functions. A degree-d Bernstein polynomial is defined explicitly by

$$B_{i,d}(\xi) = \binom{d}{i} \xi^i (1-\xi)^{d-i}, \quad \xi \in [0, 1], \tag{1}$$

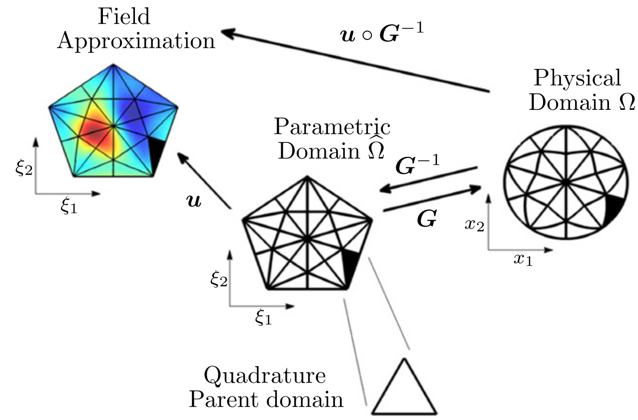

Fig. 1. Isogeometric analysis on triangulations.

where ξ is the parameter. A degree-d Bézier curve is defined in terms of $d + 1$ Bernstein basis functions and the corresponding control points $\mathbf{p}_i = (x_{1i}, x_{2i})$ as

$$\mathbf{c}(\xi) = \sum_{i=0}^{d} \mathbf{p}_i B_{i,d}(\xi). \tag{2}$$

A NURBS curve of degree-d is defined as follows

$$\mathbf{c}(\xi) = \frac{\sum_{i=0}^{n} N_{i,d}(\xi) w_i \mathbf{p}_i}{\sum_{j=0}^{n} N_{j,d}(\xi) w_j}, \tag{3}$$

where $\{\mathbf{p}_i\} = (x_{i_1}, x_{i_2})$ represents the coordinate positions of a set of $i = 0, \ldots, n$ control points, $\{w_i\}$ is the corresponding weight, and $\{N_{i,d}\}$ is the degree-$d$$B$-spline basis function, defined by a knot vector $\Xi = \{\xi_0, \xi_1, \ldots, \xi_{n+d+1}\}$. Through repeated knot insertion, the Bézier representation for each knot span of a NURBS curve can be obtained.

2.2. Bézier triangles

Bézier triangles are based on bivariate Bernstein polynomials. Let a triangle τ with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ and the barycentric coordinate of a point $\boldsymbol{\xi} \in \mathbb{R}^2$ with respect to the triangle be $\{\gamma_1, \gamma_2, \gamma_3\}$. A degree-$d$ bivariate Bernstein polynomial is defined as

$$B_{\mathbf{i},d}(\boldsymbol{\xi}) = \frac{d!}{i!j!k!} \gamma_1^i \gamma_2^j \gamma_3^k; \quad |\mathbf{i}| = d, \tag{4}$$

where $\mathbf{i}$ represents a triple index (i, j, k). A triangular Bézier patch is defined as

$$\mathbf{b}(\boldsymbol{\xi}) = \sum_{i+j+k=d} B_{\mathbf{i},d}(\boldsymbol{\xi}) \mathbf{p}_{\mathbf{i}}, \tag{5}$$

with $\mathbf{p}_{\mathbf{i}}$ represents a triangular array of control points. A rational Bézier triangle can be defined as

$$\mathbf{b}(\boldsymbol{\xi}) = \frac{\sum_{|\mathbf{i}|=d} w_{\mathbf{i}} \mathbf{p}_{\mathbf{i}} B_{\mathbf{i},d}(\boldsymbol{\xi})}{\sum_{|\mathbf{i}|=d} w_{\mathbf{i}} B_{\mathbf{i},d}(\boldsymbol{\xi})}, \tag{6}$$

where $w_{\mathbf{i}}$ are the weights associated with the control points $\mathbf{p}_{\mathbf{i}}$.

The bivariate Bernstein polynomials can be used to define a polynomial function $f(\boldsymbol{\xi})$ of degree-d over the triangle $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as:

$$f(\boldsymbol{\xi}) = \sum_{i+j+k=d} b_{ijk} B_{ijk,d}(\boldsymbol{\gamma}) \tag{7}$$