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Local computation of curve interpolation knots with quadratic precision

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ABSTRACT

There are several prevailing methods for selecting knots for curve interpolation. A desirable criterion for knot selection is whether the knots can assist an interpolation scheme to achieve the reproduction of polynomial curves of certain degree if the data points to be interpolated are taken from such a curve. For example, if the data points are sampled from an underlying quadratic polynomial curve, one would wish to have the knots selected such that the resulting interpolation curve reproduces the underlying quadratic curve; in this case, the knot selection scheme is said to have quadratic precision. In this paper, we propose a local method for determining knots with quadratic precision. This method improves on our previous method that entails the solution of a global equation to produce a knot sequence with quadratic precision. We show that this new knot selection scheme results in better interpolation error than other existing methods, including the chord-length method, the centripetal method and Foley's method, which do not possess quadratic precision.

1. Introduction

The problem of computing parametric interpolation curves is of fundamental importance in computer aided geometric design, scientific computing and computer graphics. Given a sequence of data points P_i, $i = 1, 2, \ldots, n$, an interpolation scheme needs the so called *knots* t_i associated with the P_i to produce an interpolation curve $P(t)$ with $P(t_i) = P_i$. The quality of the interpolation curve, in terms of fairness and interpolation error, depends on not only the particular interpolation scheme used, but also the selection of the knots t_i. This paper addresses the problem of computing knots for a given set of data points.

There are several existing methods for solving this problem. It is well known that using a uniform parameterization (that is, the knots t_i are equally spaced) to choose knots generally leads to unsatisfactory results when the distances between consecutive data points vary greatly. The chord length parameterization is a widely accepted method for determining knots [1–6]. This method produces satisfactory results because the accumulated chord length is a reasonable approximation to the accumulated arc length. The quality of chordal parameterization is discussed recently in paper [7]. Two other commonly used methods are Foley's method [8] and the centripetal method [9], which are the variations of the chord length method. Our experiments in Section 5 show that, in terms of interpolation error, none of these methods has a distinct advantage over the others. Moreover, in some cases, none of these methods can produce a satisfactory result.

The problem of determining knots for constructing *B*-spline/NURBS curve is discussed [10,11], where the knots are determined using an energy-optimization method. Other recent methods of determining knots can be found in [12–14].

One property shared by many of the above methods for knot selection is *linear precision*, which means that, roughly speaking, using the knots provided by such a method, the resulting interpolation curve will be a linearly parameterized straight line if the data points are sampled from a straight line. By approximation theory, in general, a smooth function with bounded derivatives can be better approximated with a polynomial of higher degree. This means that a higher order precision would in general lead to an interpolation curve with a smaller interpolation error. Our contribution is a new, local knot selection method with quadratic precision.

Since the notion of quadratic precision is central to our method, it deserves some elaboration. Suppose that we have a curve interpolation scheme that takes in a set of data points $P_i = (x_i, y_i)$ and knots t_i to produce polynomial functions $x(t)$ and $y(t)$ that form a parametric curve $P(t) = (x(t), y(t))$ to interpolate the points $P_i = (x_i, y_i)$; that is, $x(t_i) = x_i$ and $y(t_i) = y_i$, $i = 1, 2, \ldots, n$. Now consider two arbitrary quadratic functions $g(t)$ and $h(t)$. Suppose that the coordinates $\{x_i\}$ and $\{y_i\}$ of the data points P_i are sampled from $g(t)$ and $h(t)$ with the same variable values t_i in the

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increasing order; that is, $g(t_i) = x_i$ and $h(t_i) = y_i,\ i = 1, 2, \ldots, n$. Then the interpolation scheme is said to have *quadratic functional precision* if, with the same t_i as knots, it produces an interpolation curve $P(t) = (x(t), y(t))$ of the points P_i such that $x(t) = g(t)$ and $y(t) = h(t)$. That is to say, the original curve $G(t) = (g(t), h(t))$ on which the data points lie is reproduced by the interpolation scheme.

However, in a curve interpolation problem, only the data points are specified and the knots t_i are not provided as part of the input; they need to be estimated by some knot selection method before applying a curve interpolation scheme. Hence, even when an interpolation scheme possessing the quadratic functional precision is used and the data points are sampled from a quadratic polynomial curve $G(t)$, without an appropriate set of knots t_i, the resulting interpolation curve $P(t)$ may still not reproduce the curve $G(t)$.

Now we give the definition of quadratic precision of a knot selection scheme.

Definition. Suppose that an interpolation scheme possessing quadratic functional precision is used to compute an interpolation curve for a set of given data points $\{P_i\},\ i = 1, 2, \ldots, n$. A method for computing knots t_i from $\{P_i\}$ is said to have quadratic precision if, for any set of data points $\{P_i\}$ sampled from any quadratic curve $G(t)$, it produces the knots t_i such that with these knots the interpolation scheme reproduces $G(t)$ as the interpolation curve.

The first author of the present paper and his co-workers propose a method, to be referred to as the ZCM method [15, 16], that solves a global equation to find knots with quadratic precision. Here we preset a new, local method that computes knots with quadratic precision without having to solve a global equation. In contrast, the knots computed by the chord length, Foley's method and the centripetal methods have linear precision but not quadratic precision.

The remainder of the paper is organized as follows. The idea of the new method is described in Section 2. In Section 3, we discuss how to compute the local knot sequences for each data point from its neighboring points. In Section 4, we use a normalization scheme to merge the local knot sequences into a global, consistent knot sequence with quadratic precision. The comparison of the new method with the chord length method, Foley's method and the centripetal methods is presented in Section 5, and we conclude the paper in Section 6.

2. Basic idea

The main idea of the method is as follows. We locally estimate the intervals between consecutive knots based on quadratic curves interpolating each set of four consecutive data points, assuming that such four points form a locally convex configuration (i.e. no inflection). Then these local knot intervals are registered together via a normalization scheme to determine a global knot sequence. When the data points are sampled from a quadratic curve, the quadratic curves interpolating each set of four consecutive data points become the same curve, since a quadratic curve (i.e. a parabola) is uniquely determined by four points on it. We shall show that the global knot sequence thus chosen possesses quadratic precision.

Let $P_i = (x_i, y_i),\ 1 \le i \le n$, be a set of distinct data points. Four consecutive data point $P_{i+k},\ k = 0, 1, 2, 3$, form a *convex chain* if $P_iP_{i+1}P_{i+2}P_{i+3}P_i$ is a convex polygon. For the moment, we assume that every point $P_i,\ 1 \le i \le n$, belongs to at least two convex chains. For example, in Fig. 3, the point P_i belongs to the convex chains $\{P_{i-2}, P_{i-1}, P_i, P_{i+1}\}$ and $\{P_i, P_{i+1}, P_{i+2}, P_{i+3}\}$.

Let t_i denote the knots to be assigned for the points $P_i,\ 1 \le i \le n$. Our goal is to determine the t_i in such a way that, if the P_i are

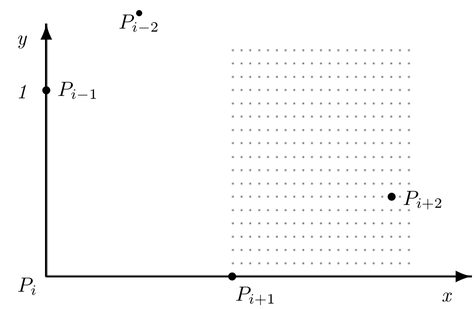

Fig. 1. Five data points.

taken from a parametric quadratic polynomial $P(\xi) = (x(\xi), y(\xi))$ defined by

$$x(\xi) = X_2\xi^2 + X_1\xi + X_0,\qquad y(\xi) = Y_2\xi^2 + Y_1\xi + Y_0 \tag{1}$$

i.e., $P_i = P(\xi_i)$, then

$$t_i = \alpha\xi_i + \beta, \quad 1 \le i \le n \tag{2}$$

for some constants α and β. This will ensure the quadratic precision, since a linear transform of the knots does not affect the type of interpolation curves produced by an interpolation scheme that has quadratic functional precision.

Suppose that the data points $P_i,\ 1 \le i \le n$, are taken from a parametric quadratic polynomial defined by Eq. (1). Then any four consecutive data points $\{P_{i-2}, P_{i-1}, P_i, P_{i+1}\},\ i = 3, 4, \ldots, n-1$ (see Fig. 1) will uniquely determine a quadratic polynomial curve $P_i(t)$ which is the same as $P(\xi)$ in Eq. (1), but possibly with a different parameterization. Since any two proper parameterizations of a quadratic curve differ by a linear reparameterization, it follows that $t = \alpha\xi + \beta$, for some constants α and β.

Let $t_j^{(0)} = \alpha_i\xi_j + \beta_i$ denote the knots computed with respect to $P_i(t)$ for the four consecutive data points $\{P_{i-2}, P_{i-1}, P_i, P_{i+1}\}$. Let $t_j^{(1)} = \alpha_{i+1}\xi_j + \beta_{i+1}$ denote the knots computed with respect to $P_{i+1}(t)$ for the four consecutive data points $\{P_{i-1}, P_i, P_{i+1}, P_{i+2}\}$. Thus, we will have two sets of knot values $t_j^{(0)}$ and $t_j^{(1)}$ for the three data points $P_j,\ j = i-1, i, i+1$, derived from the two possibly different parameterizations $P_i(t)$ and $P_{i+1}(t)$ of the same quadratic curve $P(\xi)$.

Since the two sequences of knots $t_j^{(0)}$ and $t_j^{(1)},\ j = i-1, i, i+1$, are both linearly related to ξ_i, it is possible to use a linear mapping to match up the two sequences. This is, in fact, the key idea that enables us to compute knots with quadratic precision using only local computation. At the overall level of the algorithm, suppose that we want to compute a global sequence of knots for the data points $P_i,\ i = 3, 4, \ldots, n-1$, that are taken from the same quadratic curve. We first consider all groups of four consecutive data points and compute locally the knots of the four points in each group with respect to the quadratic curve locally determined by these four points; thus each group of points will have its knot sequence of length 4. Since any two adjacent groups share three common data points and the two quadratic curves determined respectively by the two groups of points are the same curve, we can merge their knot sequences using a linear reparameterization to form a longer knot sequence.

To develop a complete solution based on this idea, we face two tasks: (1) computing the local knot sequence t_j from each group of four consecutive data points; (2) merging all these local knot sequences into a global knot sequence that has quadratic precision. These two steps will be explained in the following sections.