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G^2 surface modeling using minimal mean-curvature-variation flow[☆]

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Abstract

Physics and geometry based variational techniques for surface construction have been shown to be advanced methods for designing high quality surfaces in the fields of CAD and CAGD. In this paper, we derive an Euler–Lagrange equation from a geometric invariant curvature integral functional—the integral about the mean curvature gradient. Using this Euler–Lagrange equation, we construct a sixth-order geometric flow, which is solved numerically by a divided-difference-like method. We apply our equation to solving several surface modeling problems, including surface blending, N-sided hole filling and point interpolating, with G^2 continuity. The illustrative examples provided show that this sixth-order flow yields high quality surfaces.

Keywords: Euler–Lagrange equation; Minimal mean-curvature-variation flow; Energy functional; Discretization; Surface modeling

1. Introduction

Variational surface design. Surface fairing (see [12] for the basics on the subject), free-from surface design (see [25]), surface blending (see [27] and Fig. 5.3) and N-sided hole filling (see [31] and Fig. 1.1) have been important issues in the areas of CAD and CAGD. These problems can be efficiently solved by an energy-based variational approach (e.g. [2,4,5,10,11,21,22,24]). Roughly speaking, the variational approach is to pursue a curve or surface which minimizes certain type of energy simultaneously satisfying prerequisite boundary conditions. A problem one meets within this approach is the choice of energy models. Energy models previously used can be classified into the categories of physics-based and geometry-based. The class of physical models encompasses the membrane energy $\mathscr{E}_1$ and strain energy $\mathscr{E}_2$ of a thin elastic plate (see [7,24]):

$$\mathscr{E}_1(f) := \int_\Omega (f_x^2 + f_y^2)\mathrm{d}x\mathrm{d}y, \tag{1.1}$$

$$\mathscr{E}_2(f) := \int_\Omega (f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2)\mathrm{d}x\mathrm{d}y, \tag{1.2}$$

where $f(x, y)$ and Ω are the surface parametrization and its domain, respectively. These energies are generalized as

$$\mathscr{E}_3(\mathcal{M}) := \int_\Omega (\alpha_{11}\mathbf{r}_u^2 + 2\alpha_{12}\mathbf{r}_u\mathbf{r}_v + \alpha_{22}\mathbf{r}_v^2)\mathrm{d}u\mathrm{d}v + \int_\Omega (\beta_{11}\mathbf{r}_{uu}^2 + 2\beta_{12}\mathbf{r}_{uv}^2 + \beta_{22}\mathbf{r}_{vv}^2 - 2\mathbf{rg})\mathrm{d}u\mathrm{d}v$$

by Terzopoulos et al. in [23] for a parametric surface $\mathcal{M} := \{\mathbf{r}(u, v); (u, v) \in \Omega\}$, which can be regarded as a combination of $\mathscr{E}_1(f)$ and $\mathscr{E}_2(f)$, where $\alpha, \beta, \mathbf{g}(u, v)$ are given parameters and a vector-valued function. Recently, energy functionals based on geometric invariants begin to lead in this field. As is well-known, the area functional and total curvature functional (see [13])

$$\mathscr{E}_4(\mathcal{M}) := \int_\mathcal{M} \mathrm{d}A, \qquad \mathscr{E}_5(\mathcal{M}) := \int_\mathcal{M} (k_1^2 + k_2^2)\mathrm{d}A$$

are the most frequently used energies, where k_1 and k_2 are the principal curvatures. The minimizing surfaces of $\mathscr{E}_4(\mathcal{M})$ and $\mathscr{E}_5(\mathcal{M})$ are minimal surfaces and Willmore surfaces,

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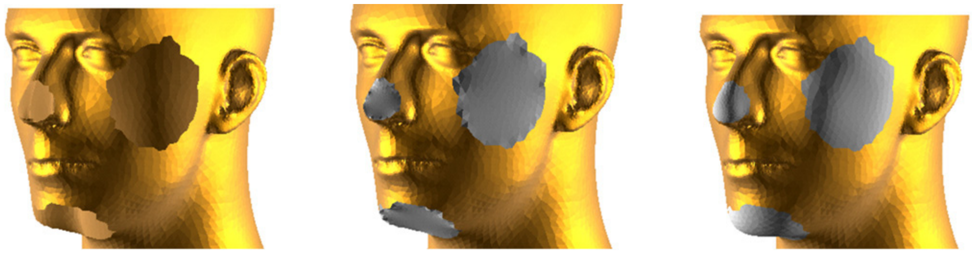

Fig. 1.1. (a) shows a head mesh with several holes. (b) shows an initial filler construction. (c) is the smooth filling surface, after 50 iterations, generated by using Eq. (3.8).

respectively. The energy

$$\mathscr{E}_6(\mathcal{M}) := \int_{\mathcal{M}} \left[\left(\frac{\mathrm{d}k_1}{\mathrm{d}\mathbf{e}_1}\right)^2 + \left(\frac{\mathrm{d}k_2}{\mathrm{d}\mathbf{e}_2}\right)^2 \right] \mathrm{d}A$$

proposed by Moreton and Séquin in [17] punishes the variation of the principal curvatures, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are principal directions corresponding to the principal curvatures k_1 and k_2.

The advantage of utilizing physics-based models is that the resulting equations are linear and therefore easy to solve. The disadvantage is that the resulting equations are parameter-dependent. That means that when a reparametrization is performed, identical surfaces may have different energies. Energy models based on geometric invariants have no such a disadvantage, and are not affected by the choice of the parametrization. For certain special cases, these two kinds of functional models are compatible when the surfaces are isometrically parameterized. For instance, $\mathscr{E}_4(\mathcal{M})$ and $\mathscr{E}_5(\mathcal{M})$ coincide with $\mathscr{E}_1(f)$ and $\mathscr{E}_2(f)$, respectively. In fact, parametrization dependent functionals can be regarded as the linear substitutes for geometric invariants. But in the general cases this equivalency is no longer correct. Another critical problem of the variational approach is how to determine the surfaces which minimize these energy functionals. Two approaches have been employed to solve this problem. One method uses the optimization approach (see [11,17,20,24]), which starts from a given surface, and searches iteratively for a next surface that has less energy. Using local interpolation or fitting, or replacing differential operators with divided difference operators, the optimization problems are discretized to arrive at finite dimensional linear or nonlinear systems. Approximate solutions are then obtained by solving the constructed systems. Another widely accepted method is based on the variational calculus. The first step of this method is to calculate the Euler–Lagrange equations for the energy functionals, and then these equations are solved for the ultimate surfaces. This method is superior to the optimization technique in general, because optimization is lack of local shape control, and computationally expensive.

Gradient descent flow method. Generally speaking, the Euler–Lagrange equations of geometric energy functionals are highly nonlinear. Except for a very limited number of simple cases where these equations do give analytic and simple solutions, directly solving the equations is difficult. The gradient descent flow method is therefore introduced to circumvent this problem. For instance, from the Euler–Lagrange equation $H = 0$ of $\mathscr{E}_4(\mathcal{M})$, which is also the definition of the so-called *minimal surface* that has been investigated for the past 250 years, we can construct a flow, called the mean curvature flow, $\frac{\partial \mathbf{r}}{\partial t} = H\mathbf{n}$. Here $\mathbf{n}$ is the normal vector field of the surface, the auxiliary variable t represents a time-marching parameter. When a steady state of the flow is achieved, we obtain $H = 0$. Similarly, for *Willmore surfaces* (see [26]) as well, the solution to the Euler–Lagrange equation $\Delta H + 2H(H^2 - K) = 0$ of the energy

$$\mathscr{E}_7(\mathcal{M}) := \int_{\mathcal{M}} H^2 \mathrm{d}A, \tag{1.3}$$

can be constructed by this gradient descent flow method. Note that functional (1.3) is equivalent to $\mathscr{E}_5(\mathcal{M})$ with the prerequisite that Gauss–Bonnet–Chern formula has been taken into account. For the purpose of volume-preserving for closed surfaces, surface diffusion flow (see [16]) $\frac{\partial \mathbf{r}}{\partial t} = \Delta H\mathbf{n}$ is sometimes employed, which can be regarded as a simplified version of the Willmore flow.

Continuity. It is well known that the second-order flows, such as the mean curvature flow or averaged mean curvature flow (see [8]), yield G^0 continuous surfaces at the boundaries of the constructed surfaces. Fourth-order flows, such as surface diffusion flow and Willmore flow ([14,26]), result in G^1 continuity. However, a higher order continuity is sometimes required in the industrial and engineering applications. For instance, in the shape design of the streamlined surfaces of aircraft, ships and cars, G^2 continuous surfaces are crucial. Therefore, higher order flows need to be considered. On this aspect, Xu et al. have utilized a sixth-order flow in [31] to achieve G^2 continuity and Zhang et al. have used another sixth-order PDE in [32,33] to obtain C^2 continuity. A sixth-order equation is also proposed in [3] by Botsch and Kobbelt to conduct real-time freeform modeling. But all these sixth-order flows and PDEs are neither physics-based nor geometry-based in the sense mentioned above.

Our contributions. In this paper, a sixth-order geometry-based PDE is introduced. It is derived from the Euler–Lagrange equation of the energy functional

$$\mathscr{F}(\mathcal{M}) := \int_{\mathcal{M}} \|\nabla H\|^2 \mathrm{d}A, \tag{1.4}$$

which punishes the total variation of mean curvature. This functional is similar to but different from the functional