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Cubic polynomial patches through geodesics

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Abstract

We consider patches that contain any given 3D polynomial curve as a pregeodesic (i.e. geodesic up to reparametrization). A curve is a pregeodesic if and only if its rectifying plane coincides with the tangent plane to the surface, we use this fact to construct ruled cubic patches through pregeodesics and bicubic patches through pairs of pregeodesics. We also discuss the G^1 connection of $(1, k)$ patches with abutting pregeodesics.

Keywords: Geodesic; Pregeodesic; Rectifying plane; Ribbon; Ruled patch

1. Preliminaries

The main goal of this paper is to exhibit a simple method to create low degree and in particular cubic polynomial surface patches that contain given curves as geodesics, when reparameterized by arclength.

The method is based on the observation that a curve which lies on a surface is a geodesic (up to parametrization) if and only if its tangent and acceleration vectors are coplanar with the surface normal along the curve. This observation rephrases the statement of O'Neill in his book Elementary Differential Geometry [1, p. 330]: a curve has geodesic curvature zero if and only if its tangent vector and the surface tangential component of the acceleration are collinear.

The problem of finding surface patches that contain a given curve as a geodesic is considered in [2] using the Frenet frame of the curve. They write down a necessary and sufficient condition for a surface to contain a prescribed curve as a pregeodesic, but do not emphasize the case of polynomial patches. The main emphasis in applications is in the shoe industry: finding surfaces that could model the shoe piece with a prescribed girth.

The mathematical connection between girth and geodesics lies in the fact that the shoe piece is fabricated from an approximately flat sheet with a minimum of stretching, so it is nearly isometric to the plane, and hence geodesics correspond to straight lines. Further applications are in textile manufacturing. This is discussed in [2] and the references therein.

Our construction of surface patches that contain prescribed polynomial cubic Bézier curves draws on the necessary relationship between the rectifying plane and the surface tangent plane, avoiding arclength parametrization and the use of Frenet frames.

2. Pregeodesics on patches

A parametrized curve which lies on a given patch is called a geodesic (see [3]) if its acceleration (i.e. the second derivative with respect to its parameter) is orthogonal to the surface along the curve. A nice introduction to geodesics which fits the needs of this paper is given in [4]. It follows that the tangent vector to a geodesic must have constant length, hence it is parametrized by arclength (or arclength times a constant).

A curve being a geodesic depends on its shape as well as on its parametrization. While working with applications for which parametrization is not relevant, it is natural to deal with geometric conditions on the curve to be a geodesic, up to reparametrization. In [1] O'Neill uses the term pregeodesic to refer to such curves. The necessary and suffcient condition for a curve to be a pregeodesic is the coplanarity of the tangent

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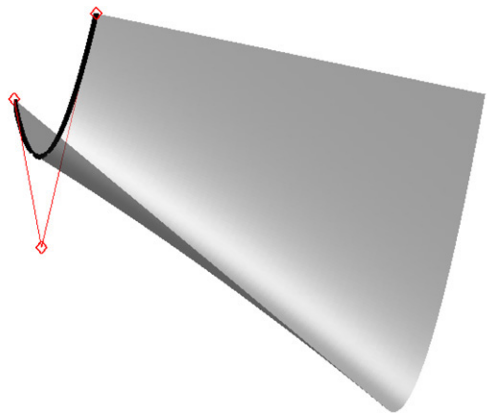

Fig. 1. Parabolic hyperboloid patch containing a parabola as pregeodesic.

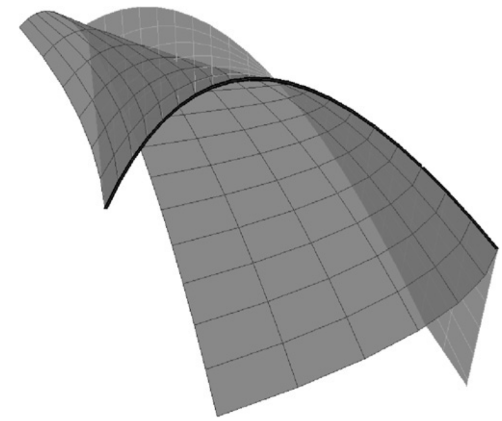

Fig. 2. Ribbon and nonruled patch through a parabola.

vector, the acceleration and the normal to the surface along the curve.

It is easy to construct families of surface patches that contain a given curve as pregeodesic. For example, if $\mathbb{x}(t)$ is any curve in 3D, such that $\mathbb{x}' \times \mathbb{x}''$ does not vanish, then for any α and $\widehat{\alpha} \neq 0$ the patch

$$\mathbb{x}(s,t) = \mathbb{x}(t) + s\{\alpha \mathbb{x}'(t) + \widehat{\alpha}\mathbb{x}'(t) \times \mathbb{x}''(t)\} \quad (1)$$

contains the isoparametric curve $\mathbb{x}(t) = \mathbb{x}(0,t)$ as pregeodesic, since the normal vector $\mathbb{n}$ to the patch is orthogonal to the plane generated by

$$\mathbb{x}'(t) \quad \text{and} \quad \alpha\mathbb{x}'(t) + \widehat{\alpha}\mathbb{x}'(t) \times \mathbb{x}''(t)$$

along the curve, so $\mathbb{n}$, $\mathbb{x}'$ and $\mathbb{x}''$ are coplanar, because $\mathbb{n} \cdot (\mathbb{x}'(t) \times \mathbb{x}''(t)) = 0$ for each t. A simple example of surfaces that contain a given curve as pregeodesic is the family of parabolic hyperboloids obtained in (1), when $\mathbb{x}(t)$ is a quadratic polynomial curve. See Fig. 1. In the special case that $\alpha = 0$ and $\widehat{\alpha} = 1$ this patch is a cylinder.

More generally any Bézier curve $\mathbb{x}(t)$ is a pregeodesic of the ruled patch given by expression (1) when we write $\alpha = \alpha(t)$ and $\widehat{\alpha} = \widehat{\alpha}(t)$, polynomials in t. This expression can also be derived (see [2]) using Frenet frames,[2] but this technique does not emphasize the construction of polynomial patches.

We refer to ruled patches as ribbons. Moreover for any arbitrary polynomial surface $\mathbb{p}(s,t)$

$$\mathbb{y}(s,t) = \mathbb{x}(s,t) + s^2\mathbb{p}(s,t)$$

is also a polynomial patch that contains $\mathbb{x}(t)$ as a pregeodesic. See Fig. 2 for an example of a ruled and a nonruled patch containing a common parabola as pregeodesic, along which both patches have coincident tangent planes. We refer to ruled patches as ribbons.

Under degenerate conditions a ribbon $\mathbb{x}(s,t)$ might fail to have a nonvanishing normal vector along $\mathbb{x}(t)$. In this case $\mathbb{x}(s,t)$ will not be a regular surface in the sense of [11]. This situation will arise if $\widehat{\alpha}(t)$, $\mathbb{x}'(t)$ or $\mathbb{x}'(t) \times \mathbb{x}''(t)$ have a zero for some t. Throughout this paper we will assume enough conditions on the curve $\mathbb{x}(t)$, its derivatives and $\widehat{\alpha}(t)$ so that the ribbons are regular surfaces (at least near the curve $\mathbb{x}(t)$), in other words that $\frac{\partial \mathbb{x}}{\partial s} \times \frac{\partial \mathbb{x}}{\partial t}$ does not vanish along $\mathbb{x}(t)$.[3]

It is clear from (1) that the degree of $\mathbb{x}(t)$ increases rapidly because of the cross-product term, even when the degrees of $\alpha(t)$ and $\widehat{\alpha}(t)$ are low. Although in the case that $\mathbb{x}(t)$ is a Bézier cubic the construction yields a ruled cubic patch if we choose α and $\widehat{\alpha}$ linear. This follows from the observation that $\mathbb{x}'(t) \times \mathbb{x}''(t)$ has degree 2.

Generally speaking, the property that guarantees that each of these patches contains $\mathbb{x}(t)$ as pregeodesic is that the tangent plane of the surface along the curve coincides with the rectifying plane of the curve.

In computer-aided geometrical design, among the ruled patches, the developable patches are especially interesting. If

$$\alpha(t) = \| \mathbb{x}'(t) \|^2 (\mathbb{x}'(t) \times \mathbb{x}''(t)) \cdot \mathbb{x}'''(t)$$
$$\widehat{\alpha}(t) = \| \mathbb{x}'(t) \times \mathbb{x}''(t) \|^2$$

the patch

$$\mathbb{x}(s,t) = \mathbb{x}(t) + s(\alpha(t)\mathbb{x}'(t) + \widehat{\alpha}(t)\mathbb{x}'(t) \times \mathbb{x}''(t))$$

is developable. For the proof it is enough to check that $\mathbb{x}'(t)$, $\mathbb{v}(t) = \alpha\mathbb{x}'(t) + \widehat{\alpha}\mathbb{x}'(t) \times \mathbb{x}''(t)$ and $\mathbb{v}'(t)$ are coplanar.

As observed in [9] developable surfaces have been widely used in engineering, design and manufacture. Also more recently [10] uses approximately developable patches to decompose meshes.

Developable patches are also important in architectural design, especially in the "paper-like" design of Frank Gehry. Beyond design, the many advantages of these surfaces in constructibility, fabrication and the coherence of physical and digital representation are studied in [6].

3. Patches through pairs of pregeodesics

Given two curves $\mathbb{x}_1(t)$ and $\mathbb{x}_2(t)$ it is possible to write down a patch $\mathbb{x}(s,t)$ such that $\mathbb{x}(0,t) = \mathbb{x}_1(t)$ and $\mathbb{x}(1,t) = \mathbb{x}_2(t)$ are pregeodesics. For this it is natural to use the cubic Hermite

[2] We thank the referee for pointing this out.

[3] The condition for regularity is that $\widehat{\alpha}(t)\mathbb{x}' \times \mathbb{x}''$ does not vanish along the curve $\mathbb{x}(t)$.