Computer-Aided Design 70 (2016) 36–45

Contents lists available at ScienceDirect

Computer-Aided Design

journal homepage: www.elsevier.com/locate/cad

Geometric characteristics of a class of cubic curves with rational offsets ☆

Xing-Jiang Lu [a], Jianmin Zheng [b,*], Yiyu Cai [c], Gang Zhao [d]

[a] Mathematics Department, Zhejiang University, Hangzhou 310027, China
[b] School of Computer Engineering, Nanyang Technological University, Singapore 639798, Singapore
[c] School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore 639798, Singapore
[d] School of Mechanical Engineering and Automation, Beihang University, Beijing 100191, China

HIGHLIGHTS

- This paper presents a geometric characterization of cubic indirect-PH curves.
- We give a construction of G^1 Hermite interpolation using indirect-PH curves.
- The optimal indirect-PH curve to G^1 Hermite interpolation is investigated.

ARTICLE INFO

Keywords:
Bézier curves
Offsets
Rational parameterization
Pythagorean hodograph
G^1 Hermite interpolation

ABSTRACT

Planar Bézier curves that have rationally parameterized offsets can be classified into two classes. The first class is composed of curves that have Pythagorean hodographs (PH) and the second class is composed of curves that do not have PHs but can have rational PHs after reparameterization by a fractional quadratic transformation. This paper reveals a geometric characterization for all properly-parameterized cubic Bézier curves in the second class. The characterization is given in terms of Bézier control polygon geometry. Based on the derived conditions, we also present a simple geometric construction of G^1 Hermite interpolation using such Bézier curves. The construction results in a one-parameter family of curves if a solution exists. We further prove that there exists a unique value of the parameter which minimizes the integral of the squared norm of the second order derivative of the curves.

1. Introduction

This paper deals with a class of Bézier curves that possess rational offset curves. Bézier curves are a common type of representations for free form curves in computer aided design and manufacturing (CAD/CAM) [1]. Offsetting has diverse engineering applications including CNC machining, motion planning, railway design, shape blending, etc. However, it is known that a Bézier curve in general may not have a rational offset [2]. This actually motivated a lot of research on offset approximation [3–8]. On the other hand, the failure of preserving rationality of the offsetting results may cause extra cost in operations involving offsets and software development as well. Therefore it is interesting to explore the curves that allow rational representation of their exact offset curves.

In 1990, Farouki and Sakkalis introduced a class of polynomial curves called Pythagorean hodograph (PH) curves [9]. A polynomial parametric curve is called a PH curve if the Euclidean norm of the hodograph of the curve is a polynomial. Hence for PH curves, their offset curves admit rational representation. Furthermore, PH curves' arc length can be expressed as a polynomial function of the parameter. Since then, many works have followed. The idea of PH curves was extended to spatial PH curves [10], rational PH curves [11,12], Minkowski PH curves [13], etc. Also many methods have been developed to perform Hermite interpolation using various PH curves [14–19]. One can easily find hundreds of papers in these topics. Detailed review on PH curves and related topics can be found in Chapter 17 of the Handbook of Computer Aided Geometric Design [1], book "Pythagorean-Hodograph Curves: Algebra

☆ This paper has been recommended for acceptance by Scott Schaefer and Charlie C.L. Wang.

* Corresponding author.
E-mail address: asjmzheng@ntu.edu.sg (J. Zheng).

http://dx.doi.org/10.1016/j.cad.2015.07.006

and Geometry Inseparable" [20], a special issue of Computer Aided Geometric Design in 2008 [21], and a recent survey [22].

Having a Pythagorean hodograph is only a sufficient condition for the offset of a polynomial curve to be rational. For example, the parabola does not belong to PH curves, but the offsets to the parabola are rational [23]. It has been proved that the necessary and sufficient condition for a properly-parameterized polynomial curve on a plane to have rational offsets is that the squared norm of its hodograph has at most two complex roots (with nonzero imaginary part) of odd multiplicity [24–26]. PH curves are a special case of such curves corresponding to the situation that the number of complex roots with nonzero imaginary part of odd multiplicity is zero. The curves in the complementary set of PH curves do not have polynomial Pythagorean hodograph, but after reparameterization via a quadratic parameter transformation, they can have a rational Pythagorean hodograph. For convenience of description, in this paper we call such curves in the complementary set *indirect-PH* curves.

Note that sophisticated algebraic methods have been employed to analyze and construct curves with rational offsets. For example, the complex variable model was used for planar PH curves [27] and indirect-PH curves [25], and the quaternion model was used for spatial PH curves [28,29]. On the other hand, polynomial curves are always representable in Bézier form and Bézier control polygons provide an intuitive way to dealing with curves. While in general the algebraic structure of the polynomial curves with rational offsets is not simply transferred to intuitive constraints on the control polygon geometry, Farouki and Sakkalis provided an elegant geometric characterization for cubic PH curves [9], which are two constraints on the lengths of legs and two interior angles of the Bézier control polygons. Wang and Fang derived the geometric characterization of quartic PH curves also in terms of Bézier control polygons [30]. This thus provides a geometric approach for constructing PH quartics. The use of control polygons was also proposed for designing planar C^2 PH quintic spline curves in [31].

In this paper, we are interested in intuitive geometric constraints for cubic Bézier curves to be indirect-PH curves based on Bézier control polygons. Except for some brief description of cubic indirect-PH curves in [25], no work has been published on the full geometric characterization of them. The main contribution of this paper is that we convert the special algebraic structure in the hodographs of indirect-PH curves into simple geometric constraints and present the full geometric characterization of cubic indirect-PH curves. The geometric characteristics are expressed using the lengths of edges and the interior angles of a quadrilateral formed from the Bézier control polygons. Furthermore, we present the construction of a characterization diagram for the Bézier control points. The idea of constructing a diagram to characterize cubic curves has been used in [32–34]. Another contribution of the paper is a simple geometric construction of G^1 Hermite interpolation using cubic indirect-PH curves. In particular, if a single cubic indirect-PH curve is available for the input Hermite data, the solution is actually a one-parameter family of curves. We prove that there exists a unique value of the parameter that minimizes the integral of the squared norm of the second order derivative of the curves.

The paper is organized as follows. Section 2 briefly reviews polynomial curves with rational offsets, which serve as the starting point of our work. Section 3 then presents the full geometric characterization of indirect-PH curves. The geometric construction of G^1 Hermite interpolation using indirect-PH curves is discussed in Section 4. Section 5 provides some examples and Section 6 concludes the paper.

2. Preliminaries

The offset to a planar curve $P(t) = (x(t), y(t))$ can be written as

$$P_o(t) = P(t) \pm d\frac{(-y'(t), x'(t))}{\sqrt{x'^2(t) + y'^2(t)}} \tag{1}$$

where d is the offset distance.

Due to the square root in the denominator, $P_o(t)$ is generally not a rational curve. In case there exists a polynomial $\sigma(t)$ such that $x'^2(t)+y'^2(t) = \sigma^2(t)$, i.e., $(x'(t), y'(t), \sigma(t))$ form a "Pythagorean triple", $P_o(t)$ is a rational curve. Such $P(t)$ is called a PH curve [9].

The rationality of the offset curve $P_o(t)$ actually depends on whether the square root has a rational parameterization. The necessary and sufficient condition for a properly-parameterized polynomial curve to have rational offsets can be described by the following theorem [25].

Theorem 1. *A properly-parameterized polynomial curve $P(t) = (x(t), y(t))$ has rationally parameterized offsets if and only if the hodograph $(x'(t), y'(t))$ has the following decomposition:*

$$x'(t) + \mathbf{i}y'(t) = \rho(t)\,[(\lambda t + 1) + \mathbf{i}\mu t]\,[u(t) + \mathbf{i}v(t)]^2 \tag{2}$$

where $\mathbf{i} = \sqrt{-1}$ is the imaginary unit, $\rho(t), u(t)$ and $v(t)$ are polynomials of t with real coefficients, λ and μ are real numbers, and $u(t)$ and $v(t)$ are relatively prime.

The decomposition (2) can be classified into two cases corresponding to the two classes of polynomial curves with rational offsets: PH curves and indirect-PH curves.

- $\mu = 0$. We have

  $$x'(t) = \rho(t)(\lambda t + 1)(u^2(t) - v^2(t)),$$
  $$y'(t) = \rho(t)(\lambda t + 1)(2u(t)v(t)).$$

  Hence $x'^2(t) + y'^2(t) = \left(\rho(t)(\lambda t + 1)(u^2(t) + v^2(t))\right)^2$, which implies that $P(t)$ is a PH curve. A linear curve (i.e., straight line segment) is a PH curve with $\lambda = 0$ and ρ, u, v being constant.
- $\mu \neq 0$. The root of $(\lambda t + 1) + \mathbf{i}\mu t = 0$ is a complex number $\frac{-\lambda+\mathbf{i}\mu}{\lambda^2+\mu^2}$, which is also a root of $x'^2(t) + y'^2(t) = 0$ with odd multiplicity. Therefore the curve $P(t)$ in this case is not a PH curve. However, we introduce a quadratic parameter transformation [25]:

  $$t(s) = \frac{B_1^2(s) + (c - 1 + b)B_2^2(s)}{(c + 1 - b)B_0^2(s) + (1 + b)B_1^2(s) + (c - 1 + b)B_2^2(s)} \tag{3}$$

  where

  $$b = \sqrt{(\lambda + 1)^2 + \mu^2}, \qquad c = \sqrt{(\lambda + 2)^2 + \mu^2},$$

  and $B_i^2(t) = \binom{2}{i}(1-t)^{2-i}t^2$ are quadratic Bernstein polynomials. It is obvious that $t(0) = 0$ and $t(1) = 1$. After the parameter transformation, the curve $P(t(s))$ has a rational Pythagorean hodograph. In fact, it is sufficient to verify that

  $$(\lambda t + 1)^2 + (\mu t)^2 = \left(\frac{(c + 1 - b)B_0^2(s) + \frac{c^2-(1-b)^2}{2}B_1^2(s) + b(c - 1 + b)B_2^2(s)}{(c + 1 - b)B_0^2(s) + (1 + b)B_1^2(s) + (c - 1 + b)B_2^2(s)}\right)^2.$$

  Therefore the curves in this class are indirect-PH curves.