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Polynomial spline interpolation of incompatible boundary conditions with a single degenerate surface☆

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HIGHLIGHTS

- Propose a method to interpolate a four-sided region with incompatible boundary.
- Achieve G^1 continuity with the boundary except for incompatible corner points.
- Utilize the property of multi-valued normal vectors at degenerate points.
- The proposed method is constructive and straightforward.

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ABSTRACT

Coons' construction generates a surface patch that interpolates four groups of specified boundary curves and the corresponding cross-boundary derivative curves. This constructive method is simple and widely used in computer aided design. However, at the corner points, it requires compatibility of the boundary conditions, which is usually difficult to satisfy in practice. In order to handle the incompatible case where the normal directions respectively indicated by two adjacent boundaries do not agree with each other at the common corner point, we utilize the property of degenerate parametric surfaces that the normal directions can converge to multiple values at degenerate points, and therefore the local degenerate geometry can satisfy conflicting conditions simultaneously. Following this idea, we use a single patch of $(2(p+2), 2(p+2))$-degree polynomial spline surface with four degenerate corners to interpolate incompatible boundary conditions, which are represented by p-degree polynomial spline curves with G^1 continuity. This method is based on symbolic operations and polynomial reparameterizations for polynomial splines, and without introducing any theoretical errors, it achieves G^1 continuity on the boundary except for the four corner points.

1. Introduction

Coons' [1] construction is a fundamental method to generate smooth transition surfaces. With four specified boundary curves and the cross-boundary derivative curves respectively corresponding to them, Coons' method generates a surface patch that interpolates all boundary conditions (see Fig. 1(a)). The generated surface is a Boolean sum of the convex combinations of two opposite boundary conditions and a correction patch spanned by the derivative conditions at the four corner points. Thanks to this simple and elegant formulation, Coons surfaces have been widely studied and used in both industry and academia during the past decades [2,3]. However, the well-known *compatibility conditions* [4–6] require that the given boundary constraints agree with each other at the corner points. These conditions usually induce extra difficulty in geometric construction as they may be too "strong" to satisfy in practice, especially in free-form surface design. For example, in Fig. 1(a), at the corner between the ith and the jth

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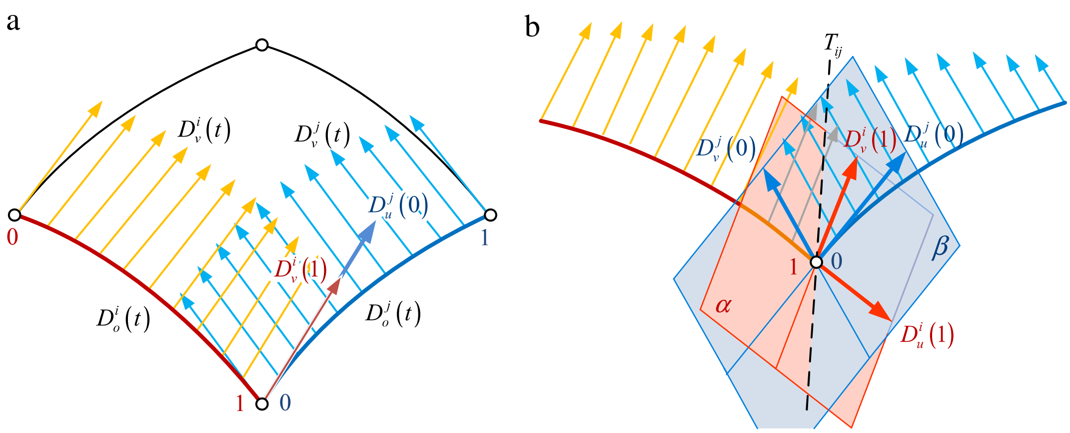

Fig. 1. Boundary conditions of Coons' construction. We denote the boundary curves by $D_o^k(t)$ ($k = 0, 1, 2, 3$) and the cross-boundary derivative curves corresponding to them by $D_v^k(t)$ ($k = 0, 1, 2, 3$). The first-order derivative curve of $D_o^k(t)$ is denoted by $D_u^k(t)$.

boundaries, the derivative of the jth boundary curve needs to agree with the cross-boundary derivative of the ith boundary. For some special incompatible cases, we can adjust the boundary condition curves to make them compatible, and the generated Coons surface can be G^1 continuous with the boundary instead of C^1 [7–10]. Nevertheless, this does not apply to all bad cases in practice. Fig. 1(b) is an example: the two groups of boundary conditions give different normal directions at the corner point, so the Coons surface should agree with both of them simultaneously if it interpolates all boundary conditions exactly. Generally, for a non-degenerate point on a surface, this conflicts with the uniqueness property of normal direction. However, Peters [11] and Reif [12] observed that the non-zero second-order derivatives define the normal direction of a point if the Jacobian matrix is zero. This property provides the possibility to satisfy conflicting boundary conditions since we can construct a degenerate corner where the normal directions converge to different values.

In this paper, we propose a method to interpolate incompatible boundary conditions represented by G^1 continuous polynomial splines curves. With p-degree input curves, the method generates a single $(2(p+2), 2(p+2))$-degree polynomial spline surface patch, which has G^1 continuity with the original condition on the boundary except for the four incompatible corner points. The method first transforms and reparameterizes all input curves to ensure that the derivatives at the end points of them are exactly zero. Then, it adjusts the magnitude of the cross-boundary derivative curves by multiplying a crescent-shaped function so that the values at the end points of these curves are zero. This reformation of the boundary conditions guarantees G^1 continuity on the boundary edge except for the two end-points, and the reformed boundary conditions fully satisfy tangent and twist compatibilities [4]. After that, it constructs the result surface according to the Boolean-sum formula of Coons' strategy. The generated surface has four degenerate corner points and exactly interpolates the specified boundary conditions except for the incompatible corner points. Our method has the following features and advantages.

- The method preserves piecewise polynomial forms. The output surface is a polynomial spline if all input curves are polynomial splines.
- Without introducing any theoretical errors, we achieve G^1 continuity with the original boundary conditions on the boundary except for the four corner points.
- The algorithm is constructive and straightforward. No iteration or large-scale matrix solving are required.

The organization of the rest paper is as follows. After reviewing related work in Section 2, we give a formal definition of the problem and the local properties of degenerate points in Section 3. Section 4 then presents the framework of our constructive method by giving the three steps to reform the boundary conditions. As this method is presented algebraically, Section 5 explains the geometric meaning of it. We give examples, comparisons and discussions in Section 6 and conclude the paper in Section 7.

2. Related work

Many boundary interpolation methods with piecewise polynomials were classified into single-patch approaches, blending approaches and splitting approaches in [13]. The compatibility problems with interpolation involve many challenges on error prevention and geometric beautification in computer aided design [14]. The difficulties of the compatibility problems were reviewed in [15]. In order to solve these problems, Gregory [4] proposed a construction method which introduces additional rational terms to prevent compatibility restrictions on the specified boundary conditions. After this method, many other interpolation schemes, for example [16–19], were proposed to adapt to different boundary conditions. For the common issue of n-sided region filling, which is a generalization of the four-sided interpolation problem, subdivision and non-polynomial blending schemes, for example, [20–23], can permit incompatible corner points as well. However, these methods only achieve smooth and visual-pleasing shapes and cannot preserve the polynomial form of the result surface. This is one of the major limitations in practice, especially for data exchange and further geometric manipulation in CAD systems.

According to the study in [24], incompatible corners can be classified into three categories. Formally, at the corner which is the common point of the ith and the jth boundary ($j = (i+1) \bmod 4$), we represent these three categories by the following four derivative vectors: $D_u^i(1)$, $D_v^i(1)$, $D_u^j(0)$ and $D_v^j(0)$ (see Fig. 1(a)).

- **Category I:** $D_u^i(1)$ and $D_v^j(0)$ are linearly dependent, and $D_v^i(1)$ and $D_u^j(0)$ are linearly dependent as well.
- **Category II:** The four derivative vectors are coplanar however cannot satisfy Category I, and the reformed boundary conditions (see the following explanation) can satisfy *twist compatibility* [1].
- **Category III:** As shown in Fig. 1(b), the four derivative vectors do not satisfy either Category I or Category II. Usually, the tangent plane spanned by $D_u^i(1)$ and $D_v^i(1)$ does not share the same