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Algebraic graph statics ☆

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HIGHLIGHTS

- General, non-procedural approach to graphical analysis of two-dimensional structures.
- Equilibrium equations derived from reciprocal relation between form and force graphs.
- States of stress of structural systems from analysis of equivalent unloaded networks.
- Construction of planar straight-line drawings of planar form graphs.
- Computational back-end for interactive graphic statics software.

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ABSTRACT

This paper presents a general, non-procedural, algebraic approach to graphical analysis of structures. Using graph theoretical properties of reciprocal graphs, the geometrical relation between the form and force diagrams used in graphic statics is written algebraically. These formulations have been found to be equivalent to the equilibrium equations used in matrix analysis of planar, self-stressed structural systems. The significance and uses of this general approach are demonstrated through several examples and it is shown that it provides a robust back-end for a real-time, interactive and flexible computational implementation of traditional graphic statics.

1. Introduction

Graphic statics is a well-known method for analysis and design of two-dimensional structures based on Cremona's extensions of Maxwell's theory of reciprocal figures [1,2]. In graphic statics, the relation between form and forces of a structural system is contained in the reciprocal relation between two diagrams. A form diagram describes the geometrical configuration of the (axial) internal and external forces of a two-dimensional structural system, and a force diagram represents their equilibrium. The combination of these two diagrams allows for an intuitive evaluation of structural behaviour, performance and efficiency at a glance. The graphical nature of the method furthermore allows for a visual verification of both the evaluation process and results [3,4], making it more transparent than arithmetic or numerical methods.

Recent developments have demonstrated how the principles of graphic statics can be combined with modern computer technologies to create interactive drawings that provide real-time, visual feedback about the relation between form and forces in response to manipulations of the drawing by the user [5–7]. It has been demonstrated that such interactive implementations are not only extremely useful for educational purposes, but also for advanced research [8]. In addition, several graphic statics tools have also been developed as plug-ins for CAD environments (e.g. [9,10]).

1.1. Problem statement

Despite its strengths, computerised (interactive) graphic statics still has some drawbacks. The process of constructing drawings can easily become tedious and time-consuming and demands a profound familiarity with the specific geometric constructions involved (e.g. [4,11]). Furthermore, since the drawings produced by the CAD tools and interactive implementations are generated in a procedural manner according to the corresponding graphic statics "recipe", they tend to be designed for specific types of structures. Modifications to the initial setup of the drawing (e.g. the number and/or connectivity of structural elements, order of the loads, …) thus require a complete redraw of the entire construction. Although the process of making a graphic statics construction is important for teaching and learning, as it helps to get familiarised with the specific geometric and structural relationships between

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different elements of such construction, it is clear that it is inconvenient for research or practical purposes.

1.2. Objectives

To fully explore the benefits of computerised graphic statics, a general, non-procedural approach is needed, which allows drawings to be created without specific knowledge of the geometric construction procedures involved.

This paper presents a robust back-end for a graphics statics application. It allows the user to start from a connected two-dimensional line drawing. The graph of this line drawing is automatically constructed and analysed to assess the feasibility of the input as a structural system or set of forces in equilibrium. If possible, a reciprocal force diagram is constructed, based on user-defined loading or self-stress conditions. The two diagrams can then be manipulated interactively without breaking their topological and geometrical relationship. As such, the user can explore different states of equilibrium by explicit, geometric modifications of the connected diagrams, or redistribution of forces within given constraints.

1.3. Contributions and outline

The remainder of this paper is organised as follows.

In Section 2, we bring together concepts and techniques from graph theory and matrix analysis of structures and present them in a unified framework for algebraic graphical analysis built around the reciprocal relation between the form and force diagrams of graphic statics.

In Section 3, we discuss a general scheme for a computational implementation of the presented approach that can be used as back-end of a real-time, interactive graphic statics application. Different steps of the implementation are illustrated using a Fink truss, which is a statically determinate structure that cannot be calculated directly with traditional graphic statics, because it contains crossing edges. Relevant algorithms are provided as code snippets.

In Section 4, the use of this framework for non-procedural graphic statics is demonstrated through four examples: a three-hinged trussed frame, an externally statically indeterminate three-bar truss, a geometrically constrained thrust line, defining its funicular loading, and a pre-stressed net. Finally, we briefly discuss the relevance of the presented approach for three-dimensional equilibrium methods, such as Thrust Network Analysis [12].

2. Theoretical framework

In this section, we describe the theoretical framework for the graph-based, algebraic approach to graphic statics presented in this paper. First, we briefly revisit traditional graphic statics, and describe the graph interpretation of form and force diagrams. Next, we formulate the reciprocal constraints between these diagrams algebraically, and derive from them the typical equilibrium equations of a (self-stressed) structural system. We furthermore show how the geometry of the force graph can be readily derived from the solution of the equilibrium equations and the topological information of the form graph. Finally, we discuss the solution strategies for different types of structural systems based on *Singular Value Decomposition* (SVD) of the equilibrium matrix of their form graph, and describe the interpretation of the obtained results in the context of graphic statics.

2.1. Graphic statics

Fig. 1 depicts a typical graphic statics drawing consisting of two diagrams that together describe the static equilibrium of a bar-node structure and a set of applied loads and reaction forces. The

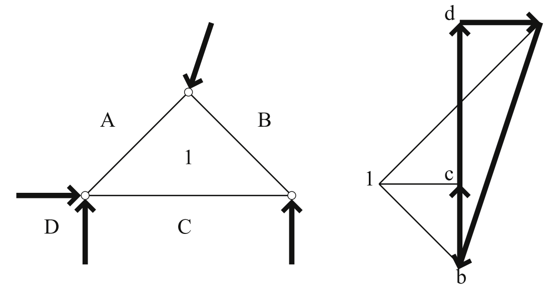

Fig. 1. Corresponding lines in reciprocal diagrams are parallel and corresponding lines which converge to a point in one diagram form a closed polygon in the other. In a graphic statics context, the diagram to the left is often called the form diagram and the one to the right the force diagram.

two diagrams are *reciprocal*: they consist of an equal number of lines, so that the corresponding lines in the two diagrams are parallel (or perpendicular, or at any constant angle), and corresponding lines which converge to a point in one diagram form a closed polygon in the other [2].

The diagram to the left is the *form diagram* and the one to the right the *force diagram*. Bow's notation is used to label spaces in the form diagram and their corresponding nodes in the force diagram [13].

A closed polygon in the force diagram represents the static equilibrium of the corresponding point in the form diagram, with the magnitude of force in the converging lines at that point equal (or proportional) to the length of the sides of the closed polygon. The form diagram thus contains the actual configuration of the bars, nodes, support forces and applied loads in space. The force diagram describes global equilibrium and the equilibrium of forces in the bars at each of the nodes. Note that it is common practice in graphic statics drawings to represent external forces in the form diagram by unit vectors, indicating only the direction and point of application of these forces. As with the internal forces in the structure, their magnitude is proportional to the length of the corresponding line segments in the force diagram.

2.2. Reciprocal graphs

The form and force diagrams can be interpreted as *directed graphs* for which (directed) *incidence* or *connectivity* matrices can be constructed describing the topological relation between branches and nodes (Fig. 2). We define the graph of the form diagram as the *form graph* G, and the one of the force diagram as its *reciprocal force graph* G^*. The force graph is the *topological dual* of the form graph with the added requirement that corresponding edges are parallel. The elements of G and G^* are *vertices*, *edges* and *faces*. Elements of the force graph are superscripted with an asterix (*).

Due to the presence of external forces in the form diagram, the form graph contains *leaf vertices*. These are vertices of degree one since they have only one connected edge, which corresponds to an external force. In Section 2.5, we will see that it is a requirement of the presented approach that the leaf vertices and their edges can be drawn on the outside of the graph, in the *outer* or *external space*. The leaf vertices will therefore be referred to as *outer* or *external vertices*; the others as *inner* or *internal*. Note that this requirement simply means that all external forces should be applied to nodes on the boundary of a structure. In Fig. 2, for example, vertices 1, 2, 4 and 6 are external vertices. They are, respectively, connected to edges 0, 1, 2 and 3 representing the external forces in the form diagram depicted in Fig. 1.

For a form graph G with e number of edges and v number of vertices, the entries of the jth column of the $[v \times e]$ connectivity