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3D hyperbolic Voronoi diagrams

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ABSTRACT

Voronoi diagrams have useful applications in various fields and are one of the most fundamental concepts in computational geometry. Although Voronoi diagrams in the plane have been studied extensively, using different notions of sites and metrics, little is known for other geometric spaces. In this paper, we are interested in the Voronoi diagram of a set of sites in the 3D hyperbolic upper half-space. We first present some introductory results in 3D hyperbolic upper half-space and then give an incremental algorithm to construct Voronoi diagram. Finally, we consider five models of 3D hyperbolic manifolds that are equivalent under isometries. By these isometries we can transform the Voronoi diagram of each model to others.

1. Introduction

Given a set of sites and a distance function from a point to a site, a *Voronoi diagram* can be roughly described as the partition of the space into cells that are the locus of points closer to a given site than to any other site.

Voronoi diagrams belong to the computational geometry's favorite structures. They arise in nature and have applications in many fields of science [1]. Excellent surveys on the background, construction and applications of Voronoi diagrams can be found in Aurenhammer's survey [2] or the book by Okabe, Boots, Sugihara and Chiu [3]. Naturally the first type of Voronoi diagrams being considered was the one for point sites and the Euclidean metric. Subsequent studies considered extended sites such as segments, lines, curved objects, convex objects, semi-algebraic sets and various distances like L_p or L_∞ [4–12]. Voronoi diagrams have interesting properties which led us to a natural question whether they will be admitted in other spaces, for example *hyperbolic surfaces*. Hyperbolic surfaces are characterized by *negative curvature*, i.e., the sum of the angles of any triangle is less than 180°. Cosmologists have suffered from a persistent misconception that negatively curved universe must be the finite hyperbolic 3D hyperbolic space [13]. Although we do not see hyperbolic surfaces around us, often nevertheless nature does possess a few. For example, lettuce leaves and marine flatworms exhibit hyperbolic geometry. There is an interesting idea about hyperbolic plane by Thurston that if we move away from a point in hyperbolic plane, the space around that point expands exponentially [14]. Hyperbolic geometry has found applications in the fields of mathematics, physics, and engineering. For example in physics, until we figure out whether or not the expansion of the universe is decelerating, hyperbolic geometry could be the most accurate way to define the geometries of fields. It has been proved that the visual information seen through our eyes and interpreted by our brain is better explained using hyperbolic geometry. Hyperbolic geometry also has practical aspects such as orbit prediction of objects within intense gravitational fields. Einstein invented his special theory of relativity based on hyperbolic geometry.

Now we switch to some applications of the Voronoi diagram in hyperbolic spaces. In [15] the authors deal with Voronoi diagram in simply connected complete manifolds with non-positive curvature, called Hadamard manifold. They proved that the facet of Voronoi diagram can be characterized by hyperbolic Voronoi diagram. Another application of Voronoi diagram in hyperbolic models is triangulating a saddle surface, which is a part of the triangulation of a general surface. On general surface, some parts have positive curvature, other parts have negative curvature and other parts near zero. In such cases, one can divide the surface into some parts, make triangulation of each part according to their curvature. Further applications of the Voronoi diagram in hyperbolic spaces is devoted to the Farey tessellation which is studied in [16]. The Teichmüller space for T^2 is the hyperbolic plane $\mathbb{H}^2 = \{z = x+iy \in \mathbb{C} | y > 0\}$: T_z^2 can be thought of as the quotient space of $\mathbb{R}^2$ over

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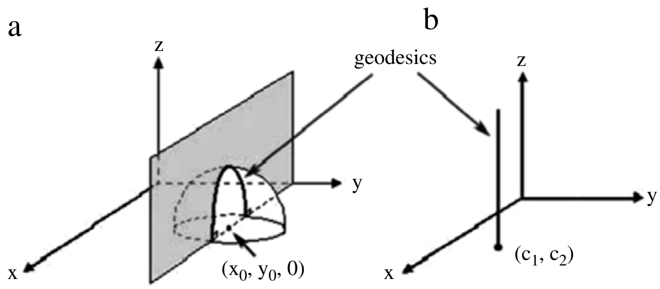

Fig. 1. Two typical geodesics of $\mathbb{H}^3$.

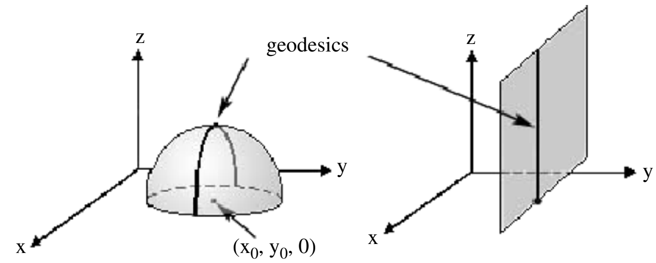

Fig. 2. Geodesics in *SH* (left) and *PH* (right).

the lattice $\{m.1 + n.z | m, n \in \mathbb{Z}\} \subset \mathbb{C}$. Let $X \subset \mathbb{H}^2$ be the set of all parameters z corresponding to the tori with three equally short shortest geodesics (i.e., tori glued from a regular hexagon). Then the Farey tessellation is nothing but the Voronoi diagram of $\mathbb{H}^2$ with respect to X.

These nice properties led us to study the Voronoi diagrams on hyperbolic spaces. In [17,18], the Voronoi diagram in upper half-plane and Poincaré hyperbolic disc, i.e., two 2D models of this space, is studied. In this paper, we generalize Voronoi diagrams in the Euclidean space $\mathbb{R}^3$ into the 3D hyperbolic upper half-space.

This paper is organized as follow. In Section 2, a brief introduction to 3D hyperbolic upper half-space and some introductory lemmata is given which will be used in the paper. Section 3 briefly reports the definition of the Voronoi diagram in 3D hyperbolic upper half-space and an incremental algorithm on construction of the Voronoi diagram in 3D hyperbolic upper half-space is given. We have implemented the algorithm and showed some examples output throughout the paper. In Section 4, we study five models for 3D hyperbolic spaces and their Voronoi diagrams. Section 5 is devoted to conclusions.

2. 3D hyperbolic upper half-space

This section deals with basic concepts of 3D hyperbolic upper half-space which are important to our approach. We also give a useful algorithm for computing the bisector between two points in this space that will be used in Section 3.

The 3*D hyperbolic upper half-space* is denoted by

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\},$$

which is a 3D Riemannian manifold with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

$\mathbb{H}^3$ is a model for 3D hyperbolic geometry and has constant negative curvature -1 (for more details see Section 6.6 of [19]).

Geodesics are basic building blocks for computational geometry on $\mathbb{H}^3$. For given two points, a geodesic (which is an analogue of a line in the Euclidean plane) is uniquely determined and is a line passing through two given points that gives the minimum length between them. The geodesics of $\mathbb{H}^3$ could be expressed in one of the following two forms (see Fig. 1):

$$\begin{cases} (x - x_0)^2 + (y - y_0)^2 + z^2 = r^2 \\ p(x - x_0) + q(y - y_0) = 0 \end{cases}$$

where $x_0, y_0, p, q, r \in \mathbb{R}$, or

$$\begin{cases} x = c_1 \\ y = c_2 \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$.

Definition 1. All the planes or spheres with equations $px + qy = r$ or $(x - x_0)^2 + (y - y_0)^2 + z^2 = r^2$ are called *hyperplane* in $\mathbb{H}^3$. We denote them by *PH* and *SH* respectively.

Remark 1. Since the rank of hyperplanes' equations is one, hyperplanes are 2D Riemannian submanifolds of $\mathbb{H}^3$.

Definition 2. A submanifold N of a Riemannian manifold M is called *totally geodesic at a point* x of N if the geodesic $\gamma(t)$ at point $x \in N$ in any neighborhood of x be a local geodesic of M and N is called a *totally geodesic submanifold* of M if it is totally geodesic at every point [20].

Lemma 1. *Hyperplanes are totally geodesic submanifolds of* $\mathbb{H}^3$.

Proof. See Fig. 2 and [20] for more details. □

Lemma 2. *Each hyperplane divides* $\mathbb{H}^3$ *into two convex regions.*

Proof. In the case of *PH*, these regions satisfy inequalities $px+qy > r$ or $px+qy < r$. In the case of *SH*, these regions satisfy inequalities $(x-x_0)^2 + (y-y_0)^2 + z^2 > r^2$ or $(x-x_0)^2 + (y-y_0)^2 + z^2 < r^2$ (see Fig. 3 and [21]). □

Lemma 3. *There is a unique geodesic passing through two given points* $A, B \in \mathbb{H}^3$ *which is a half-line or a semicircle orthogonal to the xy-plane.*

Proof. Let $\overrightarrow{k} = (0, 0, 1)$. There are two cases for the position of points A and B.

Case 1. $\overrightarrow{AB} \parallel \overrightarrow{k}$. In this case the following half-line is the geodesic passing through A and B:

$$\begin{cases} x = x_A = x_B \\ y = y_A = y_B. \end{cases}$$

Case 2. $\overrightarrow{AB} \nparallel \overrightarrow{k}$. In this case there are two unique hyperplanes passing through A and B of kinds *PH* and *SH*. The intersection of these two hyperplanes is the geodesic passing through A and B where already has been mentioned (see Fig. 1(a)). □

Let $A, B \in \mathbb{H}^3$ and $\overrightarrow{k} = (0, 0, 1)$. If $\overrightarrow{AB} \parallel \overrightarrow{k}$, i.e., $A = (x, y, z_1)$ and $B = (x, y, z_2)$, by the assumption $z_2 > z_1$, then

$$d(A, B) = \ln \frac{z_2}{z_1}.$$

Otherwise, let $C(x_0, y_0, 0)$ be the center of the geodesic (circle) passing through A and B and let α and β be the angles of OA and OB with the xy-plane respectively ($\alpha < \beta$), then

$$d(A, B) = \ln \left| \frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha} \right|.$$

See Propositions 4.1 and 4.3 in [22] for details.

Definition 3. A hyperplane passing through the midpoint of $\gamma(t)$ (the geodesic passing through the given points $A, B \in \mathbb{H}^3$) which is orthogonal to $\gamma(t)$ is to be called the *perpendicular bisector plane* of A and B.

Lemma 4. *There is a unique perpendicular bisector plane for any two given points* $A, B \in \mathbb{H}^3$.