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Critical sets in discrete Morse theories: Relating Forman and piecewise-linear approaches

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ABSTRACT

Morse theory inspired several robust and well-grounded tools in discrete function analysis, geometric modeling and visualization. Such techniques need to adapt the original differential concepts of Morse theory in a discrete setting, generally using either piecewise-linear (PL) approximations or Forman's combinatorial formulation. The former carries the intuition behind Morse critical sets, while the latter avoids numerical integrations. Forman's gradients can be constructed from a scalar function using greedy strategies, although the relation with its PL gradient is not straightforward. This work relates the critical sets of both approaches. It proves that the greedy construction on two-dimensional meshes actually builds an adjacent critical cell for each PL critical vertex. Moreover, the constructed gradient is globally aligned with the PL gradient. Those results allow adapting the many works in PL Morse theory for triangulated surfaces to Forman's combinatorial setting with low algorithmic complexity.

1. Introduction

Topological methods provide robust and well-grounded techniques to analyze and process discrete geometric models (Pascucci et al., 2011). Among those, Morse theory defines a very direct framework to study scalar functions or gradient vector fields on manifolds. Morse proved that the number and nature of critical points of regular gradient fields is constrained by the manifold on which it is defined (Morse, 1925). Smale further extended those results to coherently decompose the manifold in cells of trivial dynamic under the gradient field (Smale, 1960). In practice, this theory allows to check the coherence of mesh data structures (Dong et al., 2006; Lewiner et al., 2010) and scalar functions analysis (Stander and Hart, 2005) such as physical quantities in numerical simulation (Day et al., 2009), and grounds effective methods to visualize their main features (Bremer et al., 2004; Edelsbrunner et al., 2001).

Morse's theory were originally defined for differentiable manifolds, and thus computer-aided applications rely on approximations or discrete versions of their concepts. The most straightforward approximation of a scalar function on a mesh is given by piecewise-linear (PL) functions. Banchoff (1967) proposes a definition of critical points in that setting, which has been extended to several applications, in particular for visualization (Bremer et al., 2004) and reconstruction (Sharf et al., 2007; Van Kreveld et al., 1997; Wood et al., 2004). The approximation approach has the advantage of carrying most of the intuition of the smooth setting.

Forman (1995) developed a combinatorial formulation that completely extends Morse results to general cell complexes. This approach fits directly into the polygonal mesh setting, which is common in modeling and graphics. However, Forman's

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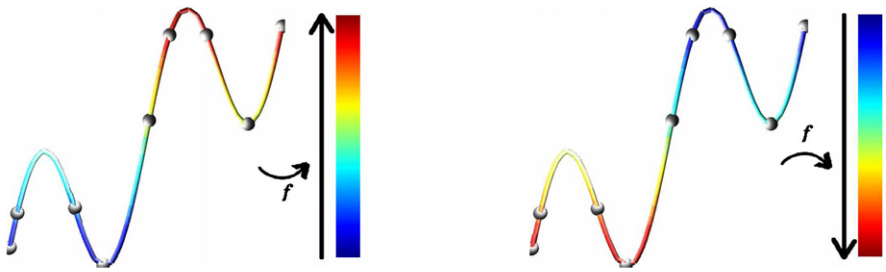

Fig. 1. A discrete curve with a scalar function f defined as the vertical projection. Forman's restriction on the dimension of critical cells may lead to non-intuitive position of critical cells. (Left) The complex is well adapted to f: the minima occur on vertices and almost all the maxima on edges. (Right) The complex is not adapted to f: the smooth minima occur out of the vertices, and critical vertices cannot coincide with them.

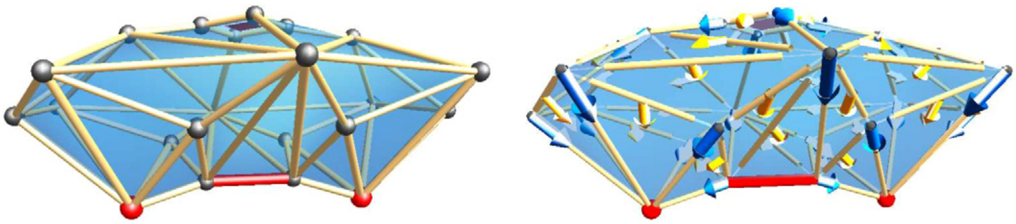

Fig. 2. Critical cells obtained from the greedy construction (there is a critical face at the top), with f being the vertical projection (left). The direction of the gradient is also coherent (right).

definitions are rather combinatoric than geometric, which complicates the construction of a discrete vector field $\mathcal{V}$ directly from a smooth function f. In particular, a cell in Forman's theory should contain points of similar dynamics under the vector field, and its dimension must represent the nature of the critical element. For example, the minima of f must be located at vertices, which in extreme cases may turn the positioning of critical cells non-intuitive (Fig. 1).

Contributions In this paper, we study the properties of greedy constructions of Forman's discrete gradient vector field from scalar functions defined on the mesh vertices (Cazals et al., 2003; Lewiner et al., 2004) as compared to PL approximations. We prove that, under regularity restrictions, Forman's critical cells defined by the greedy construction are adjacent to Banchoff's PL critical vertices (Fig. 2). We further prove that the discrete gradient globally follows the directions of $-\nabla f$. Those results open several computer applications of Morse theory currently based on PL approximations to Forman's approach, which would improve on simplicity and numerical stability while keeping performance. In particular, we mention the discrete Morse–Smale decomposition, persistence computation and Reeb graph construction.

Related work Banchoff's definition of PL critical points (Banchoff, 1967; Rourke and Sanderson, 1972) brought Morse theory to several applications, among which the effective construction of contour trees (Carr et al., 2000; Patanè et al., 2009), persistence (Cohen-Steiner et al., 2005; Edelsbrunner et al., 2000) and visualization (Fomenko and Kunii, 1997). There have been many works on PL Morse–Smale decomposition (Edelsbrunner et al., 2003, 2001) with improvements for visualization (Bremer et al., 2004; Natarajan and Pascucci, 2005).

Regarding Forman's approach, early works propose to construct discrete Morse functions with as few critical cells as possible: using lexicographic orderings (Babson and Hersh, 2005), greedy strategies (Lewiner et al., 2003a, 2004) and integer programming (Joswig and Pfetsch, 2005). Then, real applications pushed for the construction of discrete Morse function representing a sampled scalar function, using greedy strategies (Cazals et al., 2003; Lewiner, 2005) of local cancellations or successive, non-local cancellations in particular for volumetric data (Gyulassy et al., 2008, 2007). More recently, weighted graph matching approaches cleverly approximate vector fields in multi-resolution (Reininghaus et al., 2010; Reininghaus and Hotz, 2009; Reininghaus et al., 2011).

The present work uses a greedy strategy similar to (Lewiner, 2002; Lewiner et al., 2004), which is already used in several applications (Cazals et al., 2003; Weinkauf et al., 2010; Weinkauf and Günther, 2009). To our knowledge, no guarantee on the critical cells position has been presented yet.

Notation For cell complexes, we follow Forman's notation (1995): A cell complex K is a coherent collection of cells. We focus on surfaces, so the dimension p of cell σ^p is 0, 1 and 2 for vertices, edges and faces respectively. The incidence relation is denoted by $\sigma^p \succ \tau^{p-1}$. The scalar function f is defined on the set of vertices K_0, with real values. We consider here K as a triangulated manifold without boundary, although Forman's results apply to more general complexes.