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Planar C^1 Hermite interpolation with uniform and non-uniform TC-biarcs

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ABSTRACT

Pythagorean hodograph curves (shortly PH curves), introduced in Farouki and Sakkalis (1990), form an important subclass of polynomial parametric curves and currently represent standard objects in geometric modelling. In this paper, we focus on Tschirnhausen cubic as the only one Pythagorean hodograph cubic and we study planar C^1 Hermite interpolation with two arcs of Tschirnhausen cubic joined with C^1 continuity (the so-called TC-biarc). We extend results presented in Farouki and Peters (1996) in several ways. We study an asymptotical behaviour of the conversion of an arbitrary planar curve with well defined tangent vectors everywhere to a C^1 PH cubic spline curve and we prove that the approximation order is 3. Further, we analyze the shape of TC-biarcs and provide a sufficient condition for input data guaranteeing TC-biarc without local and pairwise self-intersections. Finally, we generalize the basic uniform method to the non-uniform case, which introduces a free shape parameter, and we formulate an algorithm for a suitable choice of this shape parameter such that the corresponding non-uniform TC-biarc is without local and pairwise self-intersections (if such a parameter exists).

1. Introduction

Pythagorean hodograph curves (shortly PH curves) introduced in Farouki and Sakkalis (1990) are well established objects in current geometric modelling and they are frequently studied by researchers for more than twenty years. The main reasons of their importance from the practical point of view are that they possess rational offset curves, often used in practical applications like CNC machining, and polynomial PH curves also have (piecewise) polynomial arc-length function. As current CAD/CAM systems are usually based on the NURBS representation of curves and surfaces, whose offsets are not rational in general, the motivation for the study of PH curves is obvious.

After their introduction, the exhaustive research led to several types of generalizations. Planar polynomial PH curves were firstly generalized to spatial polynomial curves in Farouki and Sakkalis (1994). Later, Pottmann (1995) gave a description of all planar rational PH curves (and also surfaces) with the help of dual representation of curves (and surfaces). Although rational PH curves still possess rational offsets, they loose the polynomial arc-length function. Spatial rational PH curves were defined more than 15 years later in Farouki and Šír (2011). Finally, Minkowski Pythagorean hodograph curves studied in Moon (1999) and Kosinka and Lávička (2010) fulfil PH condition with respect to the Minkowski metric and are useful e.g. in the connection with medial axis transform (MAT).

One of the main streams of research related to PH curves are approximation and interpolation techniques with low degree polynomial PH curves. Firstly, Farouki and Neff focused on C^1 Hermite interpolation by PH quintics (cf. Farouki and Neff, 1995). The papers devoted to Hermite interpolation by PH cubics followed – Farouki and Peters studied G^1 and C^1 Hermite interpolation via double-Tschirnhausen cubic (cf. Farouki and Peters, 1996) and, later, Meek and Walton used

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arcs of Tschirnhausen cubic for G^1 Hermite interpolation (cf. Meek and Walton, 1997). Many other papers devoted to approximation and interpolation by PH curves followed, see e.g. Moon et al. (2001); Šír et al. (2006); Pelosi et al. (2007) and Farouki (2008) and references cited therein. Further, algorithms for Hermite interpolation with spatial PH curves were studied e.g. in Jüttler and Mäurer (1999); Farouki et al. (2003); Pelosi et al. (2005); Šír and Jüttler (2007). Interpolation techniques based on rational PH curves rarely appear – G^1 and G^2 Hermite interpolation with arcs of epicycloids and hypocycloids were introduced in Šír et al. (2010); Bastl et al. (2011), where it was also proved that all algebraic epi- and hypocycloids belong to the class of rational PH curves. Finally, Hermite interpolation techniques with Minkowski PH curves are also studied – Kosinka and Jüttler (2006) focused on G^1 Hermite interpolation with MPH cubics, whereas Kosinka and Lávička (2011) presented a general framework for Hermite interpolation by MPH curves based on the choice of a suitable Hermite interpolation by planar PH curves.

It is well-known that there exists only one polynomial PH cubic, the so-called Tschirnhausen cubic. Since C^1 Hermite interpolation by arcs of Tschirnhausen cubic is not possible in general and usually is done via PH quintics, G^1 Hermite interpolation with PH cubics is commonly studied (cf. Meek and Walton, 1997; Byrtus and Bastl, 2010; Černohorská and Šír, 2010). Nevertheless, similarly to the case of classical circular biarcs, which are used for G^1 Hermite interpolation in the plane (cf. Šír et al., 2006), the problem of C^1 Hermite interpolation can be solved by composing two arcs of Tschirnhausen cubic joined with C^1 continuity (cf. Farouki and Peters, 1996).

In this paper, we focus exactly on the last mentioned case, i.e., on planar C^1 Hermite interpolation based on composition of two arcs of Tschirnhausen cubic, which we call *TC-biarcs*, and extend the results presented in Farouki and Peters (1996). The study is motivated by the recent research of other interpolation techniques. For example, as it was mentioned above, Kosinka and Lávička (2011) presented a general framework for construction of interpolation schemes by MPH curves, which uses planar interpolation by PH curves. Nevertheless, the method is highly sensitive to the degree of PH curves used for planar interpolation, i.e., the degree of MPH interpolants grows fast with the degree of planar interpolants. For such methods, interpolation by lower degree curves, even if a composition of two arcs is necessary, can be extremely advantageous.

The main new results of this paper are:

- we show that the approximation order of C^1 Hermite interpolation with uniform (Theorem 7) and also non-uniform TC-biarcs (Theorem 13) is 3 and, moreover, the proof for the approximation order also justifies $(++)$ solution as the best solution of all existing four solutions (Corollary 9),
- we analyze the shape of TC-biarcs for "reasonable" data and present the sufficient condition for input data which ensures the existence of an interpolating TC-biarc without local and pairwise self-intersection(s) (Proposition 10),
- we generalize the basic uniform method (cf. Farouki and Peters, 1996) to the non-uniform case, which introduces a free shape parameter influencing the overall shape of an interpolant for given data and which can be used e.g. for removing self-intersection(s) of the interpolant (even in the case when PH quintic contains a self-intersection) useful for instance in the approximation of cutting tool paths (see Farouki et al., 1999; Šír and Jüttler, 2005),
- we formulate an algorithm for a suitable choice of a shape parameter such that the corresponding non-uniform TC-biarc is without local and pairwise self-intersections (if such a parameter exists).

The remainder of the paper is organized as follows. Section 2 recalls some basic facts concerning PH curves. Section 3 is devoted to a brief review of results related to G^1 Hermite interpolation by PH cubics which led to a formulation of new Theorem 2. The main new results are concentrated in Sections 4 and 5 – Section 4 presents a uniform case with the proof of the approximation order and shape analysis for "reasonable" data, Section 5 focuses on a generalization to a non-uniform case introducing free shape parameter and formulates an algorithm for an optimal choice of this shape parameter. Finally, in Section 6 we conclude the paper.

2. Preliminaries

2.1. Pythagorean hodograph curves

Pythagorean hodograph curves (abbreviated to PH curves) were introduced in Farouki and Sakkalis (1990). A parametric polynomial curve $\mathbf{r}(t) = (x(t), y(t))^\top$, $t \in I \subset \mathbb{R}$, is a curve with the *Pythagorean hodograph*, if the components of its hodograph $\mathbf{r}'(t)$ satisfy the condition

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2,$$

where $\sigma(t) \in \mathbb{R}[t]$. The main advantages of PH curves are that they possess rational offset curves and have (piecewise) polynomial arc-length function. With the help of Kubota's theorem (Kubota, 1972; Farouki and Sakkalis, 1990), hodographs of all polynomial PH curves are of the form

$$\begin{aligned} x'(t) &= w(t)\big(u(t)^2 - v(t)^2\big), \\ y'(t) &= w(t)\big(2u(t)v(t)\big) \end{aligned} \tag{1}$$