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Curvature continuous bi-4 constructions for scaffold- and sphere-like surfaces[☆]

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ABSTRACT

Scaffold surfaces bound geometric structures that have a dual characterization as a curve network and a solid. A subset of scaffold surfaces can be modeled with minimal single-valence (MSV) meshes, i.e. meshes consisting of vertices of a single irregular valence n two of which are separated by exactly one regular, 4-valent vertex. We present an algorithm for constructing piecewise bi-quartic surfaces that join with curvature continuity to form scaffold surfaces for MSV meshes, for $n = 5, \ldots, 10$. Additionally, for sphere-like meshes, we exhibit bi-quartic curvature continuous surfaces with polar parameterization.

1. Introduction

Carbon nanostructures, skeletonized solids, branching vascular trees of the circulatory system, novel weave-like 3D printed materials, scaffolds for regrowing body parts, certain self-supporting architectural or biomimetic structures share a dual characterization as curve networks and as solids. Often these structures are embedded into and interact with fields such as fluids, pressure from interspersed deformable materials, outside forces, etc. so that the smoothness of the surface of the solid representation as the interface plays an important role, e.g. to encourage or discourage attachment and growth of tissue on the skeleton solid. Curvature continuity of such scaffold surfaces is of interest for flow computations, for example when applying the 'finite volume' method. While one can model scaffold surfaces with general curvature continuous (G^2) constructions, the similarity and potentially large number of scaffold branches (Fig. 1(a)) make it worthwhile to search for particularly simple constructions of branches, with good curvature distribution but fewer pieces and/or of lower polynomial degree and complexity than general G^2 constructions.

In this paper we derive such simple building blocks of curvature continuous free-form scaffold surfaces: the most basic scaffold surfaces arise from networks that admit quad-meshing with a single repeated valence n other than valence 4 (since $n = 4$ corresponds to the tensor-product mesh). These are meshes assembled from pieces that have exactly n quadrilaterals clustered around each n-valent point (Figs. 1(a), 2(a), see also Fig. 18 for a carbon nanostructure modeled by scaffold surface.) Surprisingly such *minimal single valence* (MSV) surfaces admit G^2 constructions using 2×2 patches of degree bi-4 (see Figs. 1(c), 2(b)), or, alternatively, just one bi-5 patch per quad. Both options will be presented and complemented by a bi-4 construction for sphere-like surfaces.

The core technical achievement in deriving these 'minimal' scaffold surfaces is to explicitly resolve the tightly inter-dependent second-order smoothness constraints – whose complexity seems to preclude an explicit solution – and to harness the remaining degrees of freedom in such a way as to obtain a good distribution of curvature over a set of challenging MSV meshes.

Overview. After a brief review of the state-of-the-art of high-quality G^2 surface constructions, Section 2 defines MSV meshes and the notation. Section 3 derives a bi-5 construction and Section 4 derives an alternative bi-4 G^2 construction. Section 5 adds spherical surfaces of degree bi-4.

1.1. Literature: curvature continuous surfaces

Multi-sided blends naturally arise when designing surfaces even of simple topology but complex geometry. Early G^2 constructions for multi-sided blends [1–5] focused on the then hard task of enforcing the formal mathematical constraints of curvature continuity. In the past decade the emphasis has shifted to achieving

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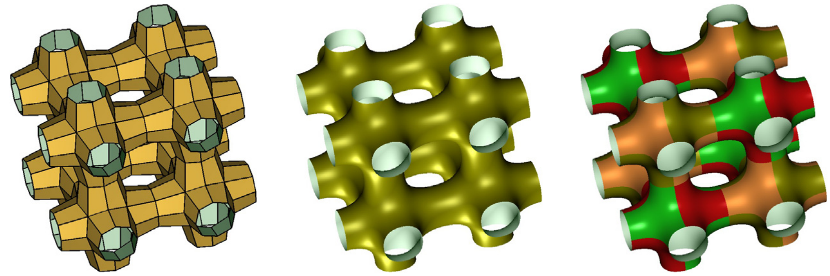
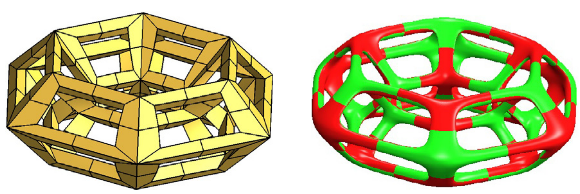


(a) MSV mesh. (b) Curvature continuous scaffold surface. (c) Neighborhoods of same valence.

Fig. 1. G^2 bi-4 scaffold surfacefrom MSV mesh with extraordinary nodes of valence 6.



(a) MSV mesh. (b) Scaffold surface.

Fig. 2. G^2 bi-4 surface of genus 25 from MSV mesh with extraordinary nodes of valence 8.

Table 1
Polynomial G^2 constructions (with good curvature distribution on challenging test data).

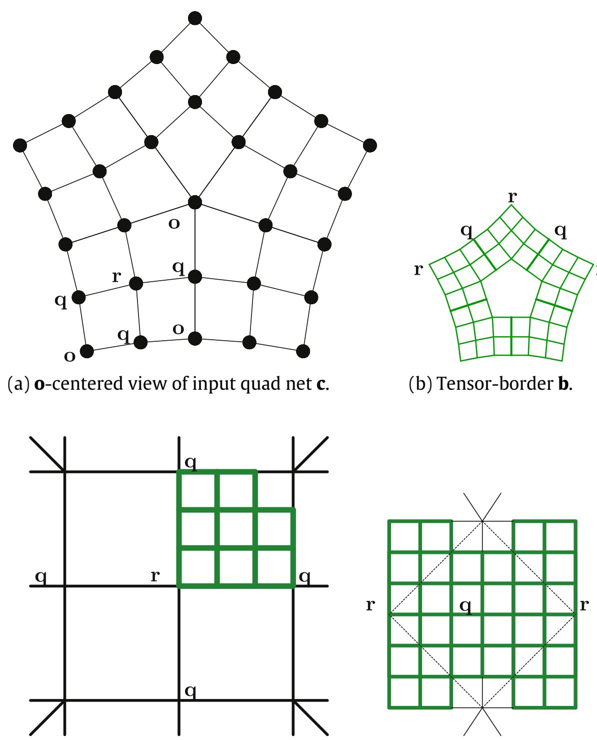
Degree	Pieces per quad	Reference
bi-9	1	[7,8]
bi-7	1	[9]
bi-6	1	[10]
bi-5	2×2	[11]

better highlight lines and distribution of curvature. Since a formal mathematical definition of good shape is illusive, highlight lines and curvature are tested on a representative set of challenging input meshes such as [6]. A consensus exists that oscillations in highlight lines and curvature distribution are to be avoided.

Table 1 summarizes recent advances in modeling high quality curvature continuous surfaces of moderate polynomial degree for a given, unrestricted quad-mesh layout. Degree bi-6 for a single patch per quad and degree bi-5 for 2×2 patches per quad are thought to be the minimal degree for obtaining good shape. By focusing on a special yet useful class of quad-meshes, the constructions in this paper further reduce the polynomial complexity of G^2 surfaces without sacrificing good shape.

2. MSV meshes and setup

In the following, we will focus on a special patchwork of quadrilateral facets (*quads*). As usual, nodes where four quads meet are called regular, while the remaining ones are called *extraordinary nodes* and are labeled **o**. For much of the exposition, we assume that all extraordinary nodes have the same valence $n > 4$ (specifically $n \in \{5, 6, 7, 8, 9, 10\}$) and each quad is uniquely associated with one extraordinary node that is one of its vertices. That is, extraordinary nodes **o** are separated by exactly one regular node of type **q** and four extraordinary nodes share one regular node of type **r** (see Fig. 3). A surface of genus g constructed with such a quad mesh has $\frac{4(2-2g)}{4-n}$ extraordinary nodes of valence n. We call this an MSV mesh (minimal single-valence mesh)—MSV meshes with $n = 4$ are tensor-product meshes, and MSV meshes with $n = 3$ form a partition of a cube



(a) **o**-centered view of input quad net **c**. (b) Tensor-border **b**. (c) **r**-centered view. (d) **q**-centered view.

Fig. 3. MSV meshes. The unlabeled vertices of the same valence n are of type **o**.

and yield sphere-like meshes. Sphere-like surfaces are discussed at the end of the paper.

We construct tensor-product patches **f** of bi-degree d in Bernstein–Bézier form (BB-form)

$$\mathbf{f}(u, v) := \sum_{i=0}^{d}\sum_{j=0}^{d} \mathbf{f}_{ij} B_i^d(u) B_j^d(v), \quad u, v \in [0, 1],$$

where $B_k^d(t)$ are the Bernstein–Bézier (BB) polynomials of degree d and $\mathbf{f}_{ij}$ are the BB-coefficients.

Interpreting the nodes of the MSV mesh as bi-cubic B-spline control points, B-spline to BB-form conversion (see e.g. [12,13]) is well-defined except for the BB-coefficients shared by $n \neq 4$ patches. The interpretation yields second-order Hermite data along and across edges between the regular points **r**, **q**. Surrounding each **o**, the Hermite data define a *tensor-border* **b** (of depth 2 and degree 3) in terms of BB-coefficients, see Fig. 3(b).

Curvature continuity is verified by relating adjacent surface pieces via a reparameterization ρ so that $\tilde{\mathbf{f}} = \mathbf{f} \circ \rho$.

Definition 1. Two surface pieces $\tilde{\mathbf{f}}$ and **f** sharing a boundary curve **e** join G^2 if there is a suitably oriented and non-singular