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Integration of generalized B-spline functions on Catmull–Clark surfaces at singularities☆

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ABSTRACT

Subdivision surfaces are a common tool in geometric modelling, especially in computer graphics and computer animation. Nowadays, this concept has become established in engineering too. The focus here is on quadrilateral control grids and generalized B-spline surfaces of Catmull–Clark subdivision type. In the classical theory, a subdivision surface is defined as the limit of the repetitive application of subdivision rules to the control grid. Based on Stam's idea, the labour-intensive process can be avoided by using a natural parameterization of the limit surface. However, the simplification is not free of defects. At singularities, the smoothness of the classically defined limit surface has been lost. This paper describes how to rescue the parameterization by using a subdivision basis function that is consistent with the classical definition, but is expensive to compute. Based on this, we introduce a characteristic subdivision finite element and use it to discretize integrals on subdivision surfaces. We show that in the integral representation the complicated parameterization reduces to a decisive factor. We compare the natural and the characteristic subdivision finite element approach solving PDEs on surfaces. As model problem we consider the mean curvature flow, whereby the computation is done on the step-by-step changing geometry.

1. Introduction

The subdivision surface concept came up with the idea of constructing smooth free-form surfaces by an iterative refinement of coarse control grids. A control grid is given by a polyhedral surface embedded in the Euclidean space $\mathbb{R}^3$. It is the most basic geometric shape representation tool in modelling and engineering systems. The refinement is done step-by-step where the repeated application of subdivision rules to the emerging grid produces finer control grids that converge towards a smooth surface, called the limit surface. A single subdivision step can be written in matrix form obtaining the so called subdivision matrix. By means of the eigendecomposition of the subdivision matrix, we are able to evaluate the limit surface in given control grid vertices. In accordance to the used subdivision rules, the emerging limit surfaces characterize different classes of surfaces. For example, Lane and Riesenfeld [1] show that using weights from Pascal's triangle produces piecewise B-spline surfaces of certain degree. However, one may have in mind that the existence of extraordinary vertices influences the smoothness of the limit surfaces. This has been extensively studied in [2,3]. Over the years, various subdivision schemes have been developed. For an overview of subdivision surfaces, we refer to Peters and Reif [4]; Cashman [5]; Ma [6].

To assemble the limit surface using subdivision might be a laborious process. By comparison, for some of the subdivision schemes, the limit surface has a piecewise parametric limit surface representation by which it can be computed in each point on the surface. In [7,8], Stam has introduced an exact evaluation scheme without any explicit subdivision of the initial control grid. Using discrete Fourier transform, an eigenstructure of the local subdivision matrix is obtained. In consequence of using Stam's idea, the labour-intensive subdivision process can be avoided, but with an undesirable side effect; the smoothness at the extraordinary vertices gets lost. Nevertheless, based on the underlying basis functions, the limit surface can be partitioned according to the control grid elements. A subdivision based finite element approach can be achieved. Such a construction has been firstly introduced to the area of engineering in [9]. Back then, it seemed to be promising. The artefacts related to the incorrect

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integration over the irregular elements may not have been fully identified.

While performing numerical experiments of the convergence of natural subdivision finite elements difficulties have been encountered in [10]. Due to the unbounded estimates over irregular elements, the reason for the errors has been indicated in the incorrect use of the Gaussian quadrature. The defective integration has been also confirmed by the studies in [11]. The reason for this is the limit surface representation for irregular elements that is given by a piecewise polynomial function. At this point, the general Gaussian quadrature is therefore an inappropriate method for an exact approximation. In order to improve this, the piecewise evaluation up to a certain subdivision depth of the elements should be performed.

1.1. Contributions

In this paper, we introduce an isoparametric subdivision finite element approach that is consistent with the classical subdivision surfaces. This means that the presented shape functions maintain the C^1-continuity at the irregular vertices. We give a precise definition of subdivision basis functions based on the Catmull–Clark subdivision. We distinguish between the natural and the characteristic parameterization of the limit surface. The last one ensures the smoothness of the classical defined subdivision surfaces, but the calculation is very expensive. We integrate the concept of element-based generating splines and isoparametric concept to obtain the corresponding finite element approaches. In addition, we derive the mass matrix and the stiffness matrix using the characteristic finite elements in greater detail. These are used for the integral representation of PDEs on subdivision surfaces. We show that the complex issue of deriving the inverse of the characteristic map reduces to an appropriate scaling factor in the integral representation.

As model problem, we investigate the mean curvature flow on closed subdivision surfaces. Therefore, the calculation is performed on a step-by-step evolving geometry. The introduced characteristic finite element improves the consistency of the subdivision control grid and, equivalently, of the limit surface. To verify this, we compare our result with the commonly used natural subdivision finite element.

1.2. Related work

Catmull–Clark subdivision surfaces [12] is one of the first and most commonly used subdivision schemes. In the limit of the subdivision, an almost everywhere C^2-continuous surface is obtained, except the finite set of extraordinary vertices; but even there the normal continuity is ensured. In [8], an efficient evaluation of the limit surface is presented. Based on this, the so called natural parameterization is defined. However, Stam's parameterization results only in C^0-continuity at the extraordinary vertices. To avoid this defect, a reparameterization based on the characteristic map can be used. Due to the computational effort, it does not seem to be feasible in practice [13]. On the other hand, the Catmull–Clark subdivision basis functions are square integrable [3] and therefore form a basis of the Sobolev space H^2. This provides a solid foundation for a finite element construction.

A finite element discretization with subdivision surfaces has been introduced in [9,14,15]. Conforming Loop subdivision finite elements on triangular meshes has been used to discretize Kirchhoff–Love's type of thin shell model. An extension of the Catmull–Clark's subdivision scheme to volumetric solids and a corresponding finite element simulation of elastic bodies has been proposed by Burkhart et al. [16]. In [17], Koiter's thin shell model has been conforming discretized using Catmull–Clark finite elements and applied to physical simulations, deformation-based modelling and calculation of free vibration modes. A numerical convergence analysis of this approach has been performed by Barendrecht [10]. Solving Poisson's equation on the disc, Nguyen et al. [11] present a classification of the Catmull–Clark finite elements according to several classical, discrete differential and isogeometric methods. In [18], adaptive isogeometric analysis is performed using truncated hierarchical Catmull–Clark subdivision splines. An isogeometric discretization approach to partial differential equations on Loop subdivision surfaces and a comparison of different quadrature schemes is discussed in [19]. Recently, Riffnaller-Schiefer et al. [20] have presented an extension subdivision based isogeometric analysis of the Kirchhoff–Love thin shell to NURBS compatible subdivision surfaces.

2. Generalized basis functions of Catmull–Clark type

One of the oldest subdivision schemes to iteratively generate smooth surfaces from coarse control grids is the Catmull–Clark scheme [12]. The scheme describes a generalization of tensor product bicubic B-splines to meshes with arbitrary topology. At each stage of the process, a control grid with quadrilateral connectivity is generated. The Catmull–Clark limit surface is an almost C^2-continuous piecewise spline surface with singularities at the extraordinary vertices, i.e. vertices with valence unequal to four. Here, a singularity is a point where the general well-behaving differentiability fails.

In [8] a stable and efficient scheme has been introduced that allows for a direct evaluation of the limit surface at any point of the domain. The limit surface can be computed elementwise without any explicit subdivision of the control grid. Elementwise means that for each element of the grid a surface patch is obtained, with a smooth transition between the patches. For this purpose, the combinatorial connectivity of the one-ring of the element has to be examined. A one-ring of an element is the union of the element and the elements sharing at least one vertex with this element. Due to the connectivity, we distinguish two types of occurring elements, regular and irregular elements, and characterize the surface patches accordingly. If the element is regular, i.e., each vertex of the element has valence four, the corresponding surface patch is a bicubic B-spline patch. Otherwise, if one of the vertices is an extraordinary vertex, the element is called irregular. In this case, the surface patch is given by an infinite sequence of nested B-spline patches. We restrict ourselves to elements with at most one extraordinary vertex.

2.1. Natural generating spline

Given an arbitrary closed control grid $\mathcal{C}_{\mathcal{Q}}$. We consider an element $Q_c \subset \mathcal{C}_{\mathcal{Q}}$ and its one-ring. Using Stam's parameterization, we are able to derive the corresponding set of element-based basis functions.

Definition 2.1 (*Natural Generating Spline*). For an element Q_c of the Catmull–Clark grid $\mathcal{C}_{\mathcal{Q}}$, we consider the set of basis functions $\left\{ b_j^* \middle| j = 0, \ldots, K-1 \right\}$. The factor $K = 2\nu + 8$, where ν denotes the valence of the extraordinary vertex, describes the size of the set of vertices in the one-ring of Q_c. Let $b(u, v)$ be a vector, where the entries $b_i,\ i = 0, \ldots, 15$ are the 16 uniform bicubic B-spline basis functions defined over the unit square $[0, 1]^2$. The natural generating spline is given by the vector

$$b^* = \left(b_0^*, b_1^*, \ldots, b_{K-1}^*\right)^T,$$