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Boundary-aware hodge decompositions for piecewise constant vector fields[☆]

Konstantin Poelke [*], Konrad Polthier

Freie Universität Berlin, Germany

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ABSTRACT

We provide a theoretical framework for discrete Hodge-type decomposition theorems of piecewise constant vector fields on simplicial surfaces with boundary that is structurally consistent with decomposition results for differential forms on smooth manifolds with boundary. In particular, we obtain a discrete Hodge–Morrey–Friedrichs decomposition with subspaces of discrete harmonic Neumann fields $\mathcal{H}_{h,N}$ and Dirichlet fields $\mathcal{H}_{h,D}$, which are representatives of absolute and relative cohomology and therefore directly linked to the underlying topology of the surface. In addition, we discretize a recent result that provides a further refinement of the spaces $\mathcal{H}_{h,N}$ and $\mathcal{H}_{h,D}$, and answer the question in which case one can hope for a complete orthogonal decomposition involving both spaces at the same time.

As applications, we present a simple strategy based on iterated L^2-projections to compute refined Hodge-type decompositions of vector fields on surfaces according to our results, which give a more detailed insight than previous decompositions. As a proof of concept, we explicitly compute harmonic basis fields for the various significant subspaces and provide exemplary decompositions for two synthetic vector fields.

1. Introduction

Hodge-type decomposition theorems form a class of central results in the study of vector fields and, more generally, differential forms on manifolds, with far-reaching applications ranging from the detection of topologically nontrivial regions and vector field analysis to the prediction of existence of solutions for PDEs. If the underlying manifold is closed, these decompositions reduce to the phrase “exact” plus “coexact” plus “harmonic”, with the space of all harmonic fields being isomorphic to a certain cohomology space—a remarkable result known as de Rham’s theorem. However, in the presence of a boundary, these decompositions become much more subtle. For instance, without any further assumptions, exact and coexact fields are no longer orthogonal to each other. In fact, they span the whole space of all fields, so there is no harmonic complement left, and one has to impose certain boundary conditions to recover the relation to the topology of the underlying geometry. Accordingly, a consistent discretization of Hodge-type theorems in the presence of a boundary is of great importance with regard to computational applications.

Piecewise constant vector fields (PCVFs) are widely used in discretization methods for differential geometric quantities on simplicial surfaces. Being defined by one tangent vector per triangle, they provide an intuitive representation for velocity and force fields in fluid dynamics or computational electromagnetics, principal curvature direction fields in shape analysis or frame fields in modeling, parametrization, and remeshing tasks in geometry processing, just to name a few examples. Moreover, they arise naturally as surface gradients of linear Lagrange elements frequently used in FEM systems, including e.g. discretizations of curvature flows or the computation of minimal surfaces. However, since a PCVF is totally discontinuous and uncoupled, the operators involved in the smooth theory do not exist for PCVFs, not even in a weak sense, and it is a priori not clear how to discretize the differentiable calculus in order to preserve the structural results of the smooth Hodge decomposition.

In this article we present a complete description of the space $\mathcal{X}_h$ of PCVFs on simplicial surfaces with boundary in terms of various Hodge-type decomposition results which are structurally consistent with their smooth counterparts in terms of topological relevance, dimension and exactness. These results give rise to a full understanding of $\mathcal{X}_h$, incorporate the sources of topological

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[*] Corresponding author.
E-mail address: konstantin.poelke@fu-berlin.de (K. Poelke).

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nontriviality that arise on surfaces with boundary – cohomology induced by boundary loops and cohomology induced by interior handles – and therefore allow for a precise characterization of PCVFs. Surprisingly, it turns out that the surface mesh has to satisfy a certain criterion for some of the discrete analogues to hold, and this criterion is not of a topological, but a combinatorial nature, i.e. depends on the triangulation, and we will give several examples below. In addition, we present a straightforward and easy to implement strategy for the computation of representative harmonic basis fields for topologically significant subspaces, and propose a refined Hodge-type decomposition method using iterated L^2-projections – two applications that are of fundamental importance in geometry processing and vector field analysis on surfaces.

Related work

Hodge-type decompositions on smooth manifolds are a classical topic, see e.g. [1]. The refinement of Dirichlet and Neumann fields into subspaces representing inner and boundary cohomology is apparently due to Hermann Gluck and Dennis DeTurck, but to the best of our knowledge first published in [2,3].

Discrete Hodge decompositions are still an active field of research, see [4] for a survey. Of the recent developments, see [5] for a decomposition in the spectral domain, Bhatia [6] for decompositions with natural boundary conditions on unbounded domains, or Ribeiro [7] for a decomposition of vector field ensembles to highlight correlations.

Piecewise constant vector fields on surfaces are present in geometric discretization schemes on surfaces at least since the work by Polthier and Preuss [8] for singularity detection and vector field decomposition, and a further investigation for closed surfaces based on this approach has been given in [9]. Previous to that, they have been used in discontinuous Galerkin methods for numerical simulations, although in this field their usage mostly restricts to problems on flat domains embedded in $\mathbb{R}^2$, which do not exhibit any interior nontrivial cohomology.

Since then there has been published an extensive amount of articles that deal with discretization schemes for vector fields and differential forms on simplicial geometries in various flavors:

Hirani [10] proposes a framework called *discrete exterior calculus (DEC)* that interprets discrete differential forms as cochains on a simplicial complex. A subset of all PCVFs (the rotation-free fields) can be regarded as closed 1-forms in this setting, but the notion of coexactness and the corresponding ansatz space differ from our approach. Hirani [11] extends this work by computing harmonic fields representing cohomology generators.

The work by Arnold [12] gives a precise numerical treatment of complexes of finite element spaces constituting a discrete de Rham complex on planar domains, focusing on stability issues for mixed problems that are deduced from a careful choice of ansatz spaces. A special case is the complex of lowest order Whitney interpolants, isomorphic to the cochain complex in DEC. Convergence estimates generalizing their results to approximating meshes are given in [13].

The paradigm of preserving the structural properties of the smooth world in the discretization is also prevalent to mimetic methods, see [14,15], in particular with respect to discretization of complexes. The idea is to discretize the operators in such a way that essential structural relations such as Green's formula are enforced to hold.

Applications of PCVFs include e.g. frame field generation and deformation of meshes with gradients of harmonic functions as in [16], field generation via quasi-harmonic potentials as in [17], or quadrilateral meshing algorithms as in [18,19].

2. Smooth decompositions

In this section we briefly summarize several smooth Hodge-type decompositions on manifolds with boundaries. For details, we refer to Schwarz [1], Abraham [20] and the survey article by Bhatia [4].

2.1. Hodge decomposition on manifolds with boundary

Starting with the physically motivated Helmholtz decomposition of a vector field inside a three-dimensional domain into a solenoidal and a conservative component in the 19th century, there is a long history of generalizations and refinements of similar decomposition statements. The *Hodge decomposition* extends the domain to arbitrary smooth manifolds M and generalizes the objects to be decomposed to differential forms, including the classical Helmholtz decomposition as the special case of 1-forms on open sets in $\mathbb{R}^3$. For closed n-dimensional smooth manifolds it states that the space of k-forms Ω^k can be L^2-orthogonally decomposed as

$$\Omega^k = \mathrm{d}\Omega^{k-1} \oplus \delta\Omega^{k+1} \oplus \mathcal{H}^k$$

where $\mathcal{H}^k$ denotes the space of *harmonic k-forms*, being simultaneously in the kernel of the exterior derivative d as well as the coderivative δ. Furthermore, the de Rham isomorphism identifies the space $\mathcal{H}^k$ as a space of representatives for the kth singular cohomology group $H^k(M)$. Here and in the following, all cohomology is understood as cohomology with real coefficients, and later on we implicitly refer to simplicial cohomology on the discretized surface, and $\oplus$ always denotes an L^2-orthogonal direct sum.

In the presence of a boundary ∂M, the spaces of exact forms $\mathrm{d}\Omega^{k-1}$ and coexact forms $\delta\Omega^{k+1}$ are no longer L^2-orthogonal to each other. To circumvent this problem one usually poses *Dirichlet boundary conditions* (the tangential part $\mathrm{tang}(\omega)$ of a differential form has to vanish along the boundary ∂M) on the space Ω^{k-1} and *Neumann boundary conditions* (the normal part $\omega|_{\partial M} - \mathrm{tang}(\omega)$ has to vanish along ∂M) on Ω^{k+1} to obtain a decomposition

$$\Omega^k = \mathrm{d}\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus \mathcal{H}^k.$$

However, in this splitting the space of harmonic forms $\mathcal{H}^k$ is now infinite-dimensional and has no topological significance anymore. Friedrichs has observed [21] that $\mathcal{H}^k$ can be further split into subspaces of *Dirichlet* and *Neumann fields*, isomorphic to relative and absolute cohomology. This leads to the *Hodge–Morrey–Friedrichs (HMF) decomposition*

$$\Omega^k = \mathrm{d}\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus \mathcal{H} \cap \mathrm{d}\Omega^{k-1} \oplus \mathcal{H}_N^k \quad (1)$$

$$= \mathrm{d}\Omega_D^{k-1} \oplus \delta\Omega_N^{k+1} \oplus \mathcal{H} \cap \delta\Omega^{k+1} \oplus \mathcal{H}_D^k \quad (2)$$

with $\mathcal{H}_N^k \cong H^k(M)$ and $\mathcal{H}_D^k \cong H^k(M, \partial M)$, where the latter denotes the kth relative cohomology space of M. Furthermore, it is

$$\mathcal{H}_N^k \cap \mathcal{H}_D^k = \{0\}, \quad (3)$$

a highly nontrivial result which can be considered an analogue of an identity-type theorem known from complex analysis (see e.g. [22]).

2.2. Inner and boundary cohomology

The decompositions in Eqs. (1) and (2) raise the question whether there is a single orthogonal decomposition involving both the spaces $\mathcal{H}_D^k$ and $\mathcal{H}_N^k$ at the same time, but this is in general not possible. Recent results [3,2] identify the obstacle as the subspace of inner cohomology representatives within $\mathcal{H}_D^k$ and $\mathcal{H}_N^k$ which