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Measure controllable volumetric mesh parameterization[☆]

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ABSTRACT

Volumetric parameterization is a fundamental problem in solid and physical modeling. In practice, it is highly desirable to control the volumes of the regions of interest in the parameter domain. This work introduces a novel volumetric parameterization method, which allows users to prescribe the target volumetric measure of the input solid.

Given a simply connected tetrahedral mesh with a single boundary surface, we first compute a volumetric harmonic map to parameterize the solid onto the unit solid ball; then we compute an optimal mass transportation map from the unit solid ball with the push-forward volume element induced by the harmonic map onto the parameter domain with the user prescribed volumetric measure. The composition of the volumetric harmonic map and the optimal mass transportation map gives a measure controllable volumetric parameterization. Furthermore, this method can handle solids with empty voids inside.

The method has solid theoretic foundation, and is based on conventional algorithms in computational geometry, and easy to implement. The experimental results demonstrate the efficiency and efficacy of the proposed method.

1. Introduction

In solid and physical modeling, volumetric parameterization plays a fundamental role, it converts unstructured tetrahedron mesh to Spline surfaces/solids.

Volumetric parameterization is essential for applications in many engineering and medicine fields, such as Spline fitting in Computer Aided Design (CAD) [1], simulation in Computer Aided Engineering (CAE), volumetric meshing in digital geometry processing [2], volumetric texture mapping in Computer Graphics, volumetric medical image registration in medical imaging [3] and so on.

Volumetric mesh parameterization refers to the process of mapping a tetrahedral mesh onto a canonical domain in three dimensional Euclidean space $\mathbb{R}^3$. The mapping is required to be homeomorphic and minimizes geometric distortions. In practice, the user prefers to allocate greater volumes in the parameter domain for the regions of interest; or the solid consists of different materials, in order to meet the accuracy requirements in the simulation/analysis process, different materials need to allocate the knots with different resolutions, therefore different parametric volumes. These requirements motivate our measure controllable volumetric parameterization.

Our method gives users the freedom to assign a target measure to the input solid. Equivalently, the user can prescribe a positive Jacobian function on the whole target domain. So that the user can enlarge some regions and shrink the others, which is valuable for Spline fitting purpose. The regions with complicated geometries or materials can be mapped to large regions in the parameter domain, and obtain more knots and control points for Splines. This helps improve the robustness and accuracy for the down stream simulation/analysis. As a special case, our method can make the whole parameterization to be volume-preserving, this is advantageous for many applications in medical imaging and visualization. Volume-preserving parameterization preserves the volumetric element, namely the Jacobian of the mapping is equal to 1 everywhere.

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In this work, we propose to use optimal mass transportation framework to achieve this goal. Given a convex domain Ω in the Euclidean space with two different measures (volume elements) μ and ν, a *Measure Preserving Map* is an automorphism of the domain itself $\varphi : \Omega \to \Omega$, and preserves the measure. Namely, for any Borel set $B \subset \Omega$, the following equality holds

$$\int_{\varphi^{-1}(B)} d\mu = \int_B d\nu.$$

The *Quadratic Transportation Cost* of the map φ is given by

$$C(\varphi) := \int_\Omega |p - \varphi(p)|^2 d\mu(p).$$

The *Optimal Mass Transportation Map* is the measure-preserving map with the least quadratic transportation cost. Optimal mass transportation guarantees the existence and the uniqueness of the map.

According to Brenier's theorem, the optimal transportation map is the gradient map of a convex function defined on the domain. The problem of finding the optimal transportation map boils down to finding the convex function. In practice, the target measure is approximated by discrete measures (Dirac measures), the convex function is approximated by the upper envelope of a family of hyper-planes in $\mathbb{R}^4$, the normals of the planes are fixed but the heights (intercepts) are unknown. The heights can be obtained by optimizing a convex energy using Newton's method.

In our current work, for a given simply connected tetrahedral mesh with a single boundary surface, we first map it onto the unit solid ball using a harmonic map; then we compute an optimal mass transportation map of the unit ball from the user prescribed volume element to the canonical Euclidean volume element. The composition of the harmonic map and the optimal mass transportation map gives the measure-controllable parameterization of the initial tetrahedral mesh. Furthermore, this method can handle solids with empty voids inside.

Contributions. This work proposes a novel algorithm to compute volumetric parameterization with a prescribed measure for a simply connected tetrahedral mesh with a single boundary surface, furthermore the method is capable of handling solids with voids inside. The algorithm is based on the recently developed discrete optimal mass transportation theory [4], which is rigorous and easy to implement. This work mainly focuses on the algorithmic design and implementation.

2. Previous works

The literature for parameterization is vast, so a thorough survey is beyond the scope of the current work. In the following, we only briefly review the most related works in volumetric parameterization and optimal mass transportation.

2.1. Volumetric parameterization

There are mainly three kinds of methods for volumetric parameterization.

The most widely used method is harmonic mapping, whose basic idea is to reduce the discrete harmonic energy by a variational procedure. This method was first proposed by Wang et al. in [3]. They mapped a genus-zero volume to a solid sphere, and later they used this method on brain mapping [5]. The harmonic mapping between two solid models was computed with a meshless approach by Li et al. [6]. Martin et al. constructed trivariate spline for cylindrical volumes by computing harmonic volumetric parameterization in [7]. Xu et al. [8] designed a bi-harmonic map applying a multiple fundamental solutions system for fast computation. Using Green's functions, Xia et al. [9] parameterized star-shaped volumes. They showed that the constructed map is bijective and smooth except at unique critical point. They also proposed an algorithm to decompose a volume into the direct product of a two-dimensional (2D) surface and a one-dimensional (1D) curve and then traced the integral curve along the harmonic function in [10]. In 2014, by combining harmonic map and streamline approach, Gupta et al. [11] presented an approach for the problem of volumetric parameterization of a general nonconvex (genus-0) domain to its topologically equivalent convex domain.

Many mappings are constructed by means of generalized barycentric coordinates with closed form expressions [12]. The mean-value coordinates method was extended from surface [13] to volume by Ju et al. [14] and Floater et al. [15] by computing the interpolation of volumetric data. Lipman et al. [16] proposed Green coordinates which lead to mappings with shape-preserving property.

Another kind of popular method is to find a mapping minimizing a specific energy. Chao et al. [17] minimized the so-called ARAP (as-rigid-as-possible) deformation energy, which is a simple geometric model measuring distance from the Jacobian of the mapping to an isometry. Frame field driven methods use the energy measuring difference between the Jacobian and the guidance frame field [18–20] and compute the volumetric parameterization in a variational way. In 2015, by deriving a 3D version of stretch-distortion energy and incorporating fixed boundary conditions, Jin et al. [21] extended the stretch-minimizing method to volumetric parameterization.

2.2. Optimal mass transportation

Monge raised the classical *Optimal Mass Transport Problem* that concerns determining the optimal way, with minimal transportation cost, to move a pile of soil from one place to another [22]. Kantorovich [23] has proved the existence and uniqueness of the optimal transport plan based on linear programming. Monge–Kantorovich optimization has been used in numerous fields from physics, econometrics to computer science including data compression and image processing [24]. Recently, researchers have realized that optimal transport could provide a powerful tool in image processing, if one could reduce its high computational cost [25,26]. However, it has one fundamental disadvantage that the number of variables is $O(k^2)$, where k is the number of discrete samples, which is unacceptable to computer vision and medical imaging applications since a high resolution 3D surface normally includes up to hundreds of thousands of vertices.

An alternative Monge–Brenier optimization scheme can significantly reduce the number of variables to be optimized. In late 1980s, Brenier [27] developed a different approach for a special class of optimal transport problems, where the cost function is a quadratic distance. Brenier's theory shows that the optimal transport map is the gradient map of a special convex function. Assume the target domain is discretized to k samples, the Monge–Brenier's approach reduces the unknown variables from $O(k^2)$ to $O(k)$, which greatly reduces the computation cost, and improves the efficiency. In our framework, we take Monge–Brenier's approach. However, our work is based on the newly discovered variational principle [4] which is the underspinning of Monge–Brenier's approach. Our framework is general and works with any valid measures, μ and ν, defined on two surfaces. Within the scope of this paper, we only consider the area induced measures. Recently, Su et al. applied Brenier's approach for shape matching and comparison in computer vision field [28]. Similar method has been used for surface area-preserving parameterization in graphics/visualization field by Kaufman et al. in [29], which focuses on the optimal mass