Computer-Aided Design 77 (2016) 86–97

Contents lists available at ScienceDirect

Computer-Aided Design

journal homepage: www.elsevier.com/locate/cad

Geometric Hermite interpolation by a family of intrinsically defined planar curves

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ARTICLE INFO

Article history:
Received 22 September 2015
Accepted 4 April 2016

Keywords:
Geometric Hermite interpolation
Intrinsic equation
Fair curves
Offset curves

ABSTRACT

This paper proposes techniques of interpolation of intrinsically defined planar curves to Hermite data. In particular, a family of planar curves corresponding to which the curvature radius functions are polynomials in terms of the tangent angle are used for the purpose. The Cartesian coordinates, the arc lengths and the offsets of this type of curves can be explicitly obtained provided that the curvature functions are known. For given G^1 or G^2 boundary data with or without prescribed arc lengths the free parameters within the curvature functions can be obtained just by solving a linear system. By choosing low order polynomials for representing the curvature radius functions, the interpolating curves can be spirals that have monotone curvatures or fair curves with small numbers of curvature extremes. Several examples of shape design or curve approximation using the proposed method are presented.

1. Introduction

Constructing fair curves that interpolate given boundary data is a fundamental task in shape design and geometric processing. Usually, G^1 Hermite interpolation is to find a planar or spatial curve between two given points, matching the corresponding unit tangent vectors at the points. If the G^1 interpolating curve also matches the given curvatures at the two ends, a G^2 Hermite interpolating curve is obtained.

A lot of work have been given in the literature to deal with the problem of G^1 or G^2 Hermite interpolation. Due to their popularity, polynomial or rational polynomial curves have been frequently used for geometric Hermite interpolation. Besides the interpolation constraints, the additional degrees of freedom of a polynomial curve can usually be determined by minimizing a bending energy function [1–5]. Alternatively, the free parameters within a polynomial or a rational curve should be determined by the criterion that the curve is a spiral [6–8] or the interpolating curve can have circle or conic precision [9,10]. When a uniform B-spline curve is used for interpolation of a sequence of sampled points, tangents and curvatures, the control points of the interpolating curve should be uniformly spaced such that the obtained curve is as fair as possible [11]. As Pythagorean Hodograph (PH) curves have rational offsets, this kind of curves has usually been used for interpolation of Hermite data for NC machining [12–15].

For applications such as fair shape design, highway route design and description of trajectories of mobile robots, various types of spirals that have monotone curvatures have been used for interpolation of G^1 Hermite data [16–20] or G^2 Hermite data [21–24]. Levien and Séquin have used Euler spiral for shape design [25] and argued some properties of a "best" interpolating spline [26]. Due to their nice properties, spiral fat arcs can even be used to find the intersections between curves [27]. Miura et al. derived a general equation of aesthetic curves, and presented their radius of curvature in terms of arc lengths [28,29]. In order to control the aesthetic curves interactively, Yoshida et al. proposed to represent the aesthetic curves with respect to the tangent angle [30]. Meek et al. derived the condition and proposed a method for planar two-point G^1 Hermite interpolation with log-aesthetic spirals [19]. For convenience of evaluation, Ziatdinov et al. derived the analytic parametric equations of log-aesthetic curves in terms of incomplete gamma functions [31]. A general family of fair curves called superspirals of which the radius of curvature is represented by Gauss hypergeometric function is given in [32]. Though superspirals have elegant properties, lack of simple analytic representation may restrict the practical applications.

Our proposed curve interpolation schemes are motivated by analytic representation of spirals and curvature plot driven curve design [33,34]. We expect that fair curves are explicitly defined by prescribed curvature functions and the obtained curves can interpolate given boundary data by choosing proper free parameters

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http://dx.doi.org/10.1016/j.cad.2016.04.001
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within the curvature functions. Based on the fact that the Cartesian coordinates of a spiral can be obtained by integrals of the radius of curvature we define a family of curves by choosing the curvature radius as polynomials. This is based on the following two observations. First, any continuous curvature radius function can be approximated by polynomials according to the Weierstrass approximation theorem. Second, the integrals that define the Cartesian coordinates of the curves can be explicitly evaluated. As a result, if the G^1 or G^2 boundary data are given, the coefficients of the polynomials can be directly derived from the interpolation equations and the interpolating curve can be explicitly obtained.

The paper is organized as follows. In Section 2, we define a family of integral curves of which the radius of curvature is represented by polynomials. Section 3 describes the planar two-point G^1 Hermite interpolation by a single integral curve with or without arc length constraint. Section 4 describes the planar two-point G^2 Hermite interpolation by a single integral curve with non-vanishing curvature radius. Section 5 is devoted to interpolation of the two-point Hermite data by two smooth-connected regular integral curves. How to interpolate a sequence of points with a G^1 or G^2 spline curve is discussed in Section 6. In Section 7, several interesting examples are given and Section 8 concludes the paper.

2. A family of intrinsically defined planar curves

In this section we revisit the definition of planar curves using curvature radius functions in terms of the tangent angle. Some nice properties of this kind of curves will be discussed. In particular, explicit formulae for computing the Cartesian coordinates will be given when the radius of curvatures are represented by polynomials.

Suppose that θ is the angle between the tangent direction of a local convex curve $\mathbf{r}(\theta)$ and the positive direction of x-axis. Then the Cartesian coordinates of the curve can be computed by the intrinsic equation [35]

$$\mathbf{r}(\theta) = \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_0^\theta \rho(t)\cos t dt \\ \displaystyle\int_0^\theta \rho(t)\sin t dt \end{pmatrix} \tag{1}$$

where $\rho(t)$ is the radius of curvature of the curve. From the equation we know that the curve $\mathbf{r}(\theta)$ passes through the origin at $\theta = 0$ and is tangent to the x-axis at the origin.

With simple evaluation we have

$$\mathbf{r}'(\theta) = \begin{pmatrix} x'(\theta) \\ y'(\theta) \end{pmatrix} = \begin{pmatrix} \rho(\theta)\cos\theta \\ \rho(\theta)\sin\theta \end{pmatrix}.$$

Then the length of the derivative is $|\mathbf{r}'(\theta)| = \rho(\theta)$ and the unit normals to the curve are $\pm(-\sin\theta, \cos\theta)^T$. If the function $\rho(\theta)$ does not vanish on the whole domain, the obtained curve $\mathbf{r}(\theta)$ is regular. Since a regular integral curve defined by Eq. (1) is locally convex, it is known that any regular curve segment $\mathbf{r}(\theta), \theta_1 \le \theta \le \theta_2$ lies in the triangle formed by $\mathbf{r}(\theta_1)$, $\mathbf{r}(\theta_2)$ and the intersection point of tangent lines passing through $\mathbf{r}(\theta_1)$ or $\mathbf{r}(\theta_2)$ when $|\theta_2 - \theta_1| < \frac{\pi}{2}$. A potential application of this property is to find the intersections between two integral curves.

The arc length of the curve defined by Eq. (1) can be obtained as

$$L(\theta) = \int_0^\theta \rho(t)dt. \tag{2}$$

Choosing the unit normal to the curve as $\mathbf{n}(\theta) = (-\sin\theta, \cos\theta)^T$, the offset curve to $\mathbf{r}(\theta)$ with a sighed distance h is obtained as

$$\mathbf{r}_{offset}(\theta) = \mathbf{r}(\theta) + h\mathbf{n}(\theta). \tag{3}$$

From Eq. (1) we know that if $\rho(t)$ is chosen some elementary functions, e.g., polynomials, exponential functions or trigonometric functions, etc., the Cartesian coordinates of the curve can be explicitly obtained. Consequently, the arc length and the offset of the curve can be explicitly obtained too. For ease of computation we choose $\rho(t)$ as polynomials in this paper. As discussed later, the curves defined in this way can be used for geometric Hermite interpolation directly.

It is clear that the integral curve $\mathbf{r}(\theta)$ is a circle or a circular arc when $\rho(t)$ equals a constant. To introduce more degrees of freedom for designing the curvature profiles or for interpolation of Hermite data we choose polynomials other than constants to represent the curvature radius of an integral curve.

Firstly, we assume that the radius of curvature is formulated as a linear function

$$\rho(\theta) = c\theta + d, \quad \theta \in [0, \phi]. \tag{4}$$

Substituting Eq. (4) into Eq. (1), we obtain the Cartesian coordinates of the curve as follows

$$\begin{cases} x(\theta) = c(\theta\sin\theta + \cos\theta - 1) + d\sin\theta \\ y(\theta) = c(-\theta\cos\theta + \sin\theta) + d(1 - \cos\theta). \end{cases} \tag{5}$$

We note that the curve $\mathbf{r}(\theta)$ just represents a circle involute when the parameter d is chosen zero [35]. By choosing proper values for both free parameters c and d, the curve $\mathbf{r}(\theta)$ will be used for interpolation of G^1 Hermite data.

Secondly, we derive the explicit representation of curve $\mathbf{r}(\theta)$ by assuming that the radius of curvature is a cubic function

$$\rho(\theta) = a\theta^3 + b\theta^2 + c\theta + d, \quad \theta \in [0, \phi]. \tag{6}$$

Substituting Eq. (6) into Eq. (1), it yields

$$\begin{cases} x(\theta) = (a\theta^3 + b\theta^2 + c\theta + d - 6a\theta - 2b)\sin\theta \\ \qquad + (3a\theta^2 + 2b\theta + c - 6a)\cos\theta + 6a - c \\ y(\theta) = (6a\theta + 2b - a\theta^3 - b\theta^2 - c\theta - d)\cos\theta \\ \qquad + (3a\theta^2 + 2b\theta + c - 6a)\sin\theta - 2b + d. \end{cases} \tag{7}$$

When all the coefficients a, b, c and d are free, the integral curve $\mathbf{r}(\theta)$ can be used for interpolation of G^2 Hermite data in a specially chosen coordinate system. If the coefficient a is set zero, the Cartesian coordinates by Eq. (7) are the coordinates of curve $\mathbf{r}(\theta)$ with a quadratic integral kernel $\rho(\theta)$. This kind of curves will be used for G^1 Hermite interpolation that has a prescribed arc length.

Fig. 1 illustrates an integral curve that is defined by a linear curvature radius function. An integral curve together with a cubic curvature radius function is illustrated in Fig. 2.

We note that PH curves have also explicit representations of arc lengths and offsets [36], but the curves defined by Eq. (1) no longer lie in the polynomial space and their offsets are not rational curves anymore. If the curvature radius function $\rho(\theta)$ is of degree n, the intrinsically defined curves and their offsets just lie in the linear space $\Omega = span\{1, \sin\theta, \cos\theta, \theta\sin\theta, \ldots, \theta^n\cos\theta\}$. Within the new space typical curves such as circle involutes can be represented exactly by the presented forms.

3. G^1 Hermite interpolation

In this section we propose algorithms for constructing integral curves as defined in Section 2 to interpolate given G^1 Hermite data. Suppose that $\{P_1, T_1; P_2, T_2\}$ are the given boundary points and the unit tangents at the points. We will construct an interpolating curve by choosing a linear curvature radius function. If the arc length of the interpolating curve has also been given, a quadratic curvature radius function will be used for the construction of the interpolating curve.