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Geometric continuity C^1G^2 of blending surfaces

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ABSTRACT

In this paper, we study the C^1G^2 continuity of surfaces by a shape-blending process. Furthermore, we study the continuity of the ruled surfaces constructed by linear interpolation between two pairs of C^1G^2 continuous curves. We give some conditions for the C^1G^2 continuity of composite surfaces in a shape-blending process. A practical approach is proposed to maintain the C^1G^2 continuity of Bézier surfaces pairs in a shape-blending process by adjusting the control points along the common boundary of the resulting surface-pair. We finish by proving and justifying the efficiency of the approaching method with some graphical examples.

1. Introduction

Geometric continuity G^n is widely recognized as the suitable way to fit or interpolate together two curves or surfaces in Computer Aided Geometric Design. For surface case, the conditions for patches to be of G^n continuity and the constructions of G^n conditions surfaces are important topics in the fields of Computer Aided Geometric Design and Computer Graphics. Linear interpolation between G^1 piecewise continuous curves may result geometrically in non-continuous curves. This is also satisfied for the ruled surfaces constructed with G^1 continuous curves.

In [1] the author describes a geometric construction which ensures tangent plane continuity between adjacent polynomial surfaces patches that allows a straight forward geometric interpolation. In [2] the author put the theory of geometric continuity to practice use by offering a number of methods for geometrically constructing spline curves possessing geometric continuity. Then, he describes how Bézier curve segments can be stretched together with G^1 or G^2 continuity, using geometric constructions, which lead to the development of geometric constructions for quadratic G^1 and cubic G^2 Beta-splines.

Bézier patches are important and useful patches in CAGD. In [3] the authors give the necessary and sufficient conditions for two Bézier patches of arbitrary degrees to be of G^n-continuity along a common boundary curve. Also, they concentrate their paper on the case where two tensor product Bézier patches join along a common boundary curve. In [4], the author provides explicit representations of both neighboring columns of control points by prescribing not only the common edge but also the position of the tangent plane in a suitable way. Its approach can be extended to the case of G^2-continuity and even to the cases of rational Bézier patches.

In [5] the author presents an approach for maintaining G^1 continuity of the blended curves by adjusting the junction points of the curves, also he studies the continuity of the ruled surfaces created by linear interpolation between two pairs of G^1 continuous curves.

In [6] we study the linear interpolation between C^1G^2 piecewise continuous curves. We establish some criteria to maintain C^1G^2 continuity for the linear interpolation constructed curves. For practice, we give an approach to maintain the C^1G^2 continuity of Bézier curves shape-blending process by adjusting the control points. In this paper, we present an extension to the surfaces case of [6]. That means we give sufficient conditions to succeed the geometric C^1G^2 continuity of surfaces and we give an approach to maintain the C^1G^2 geometric continuity by adjusting the points of control for Bézier surfaces. We finish by illustrating some graphical examples in order to show the validity in practice of this method.

2. Ruled surface C^1G^2 continuous between curves segments

Consider the linear interpolation of two composite curves P and Q, assume that both curves are composed of two curves segments $P_l(u)$, $P_r(u)$ and $Q_l(u)$, $Q_r(u)$ where $0 \le u \le 1$, respectively, so that $P_l(u)$ and $P_r(u)$ are C^1G^2 continuous at $P_l(1) = P_r(0)$, i.e. there exists $\lambda \in \mathbb{R}$ such that $P_l'(1) = P_r'(0)$ and $P_l''(1) = P_r''(0) + \lambda P_r'(0)$,

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and $Q_l(u)$ and $Q_r(u)$ are C^1G^2 continuous at $Q_l(1) = Q_r(0)$, i.e. there exists $\mu \in \mathbb{R}$ such that $Q_l'(1) = Q_r'(0)$ and $Q_l''(1) = Q_r''(0) + \mu Q_r'(0)$.

Consider a ruled surface $S_l(u, v)$ between the segments curves $P_l(u)$, $Q_l(u)$ and a ruled surface $S_r(u, v)$ between the segments curves $P_r(u)$, $Q_r(u)$, with $u, v \in [0, 1]$, so that

$$S_l(u, v) = (1 - v)P_l(u) + vQ_l(u),$$
$$S_r(u, v) = (1 - v)P_r(u) + vQ_r(u).$$

Theorem 1. *The surfaces $S_l(u, v)$ and $S_r(u, v)$ are C^1G^2 continuous along their common edges $S_l(1, v) = S_r(0, v)$, for all $v \in [0, 1]$ if $\lambda = \mu$ or $P_r(0) - Q_r(0)$, $P_r'(0)$ and $Q_r'(0)$ are coplanar.*

Proof. The surfaces $S_l(u, v)$ and $S_r(u, v)$ are C^1G^2 continuous along their common edges $S_l(1, v) = S_r(0, v)$, for all $v \in [0, 1]$ if it exists two real functions $\alpha(v)$, $\beta(v)$, $v \in [0, 1]$ such that

$$\frac{\partial^2 S_l}{\partial u^2}(1, v) = \frac{\partial^2 S_r}{\partial u^2}(0, v) + \alpha(v)\frac{\partial S_r}{\partial u}(0, v) + \beta(v)\frac{\partial S_r}{\partial v}(0, v). \quad (1)$$

This implies that

$$\begin{aligned}(1 - v)P_l''(1) + vQ_l''(1) \\ &= (1 - v)P_r''(0) + vQ_r''(0) + \alpha(v)((1 - v)P_r'(0) \\ &\quad + vQ_r'(0)) + \beta(v)(-P_r(0) + Q_r(0)).\end{aligned}$$

Then

$$\begin{aligned}&\beta_r(v)(P_r(0) - Q_r(0)) + (\lambda - \alpha(v))(1 - v)P_r'(0) \\ &+(\mu - \alpha(v))vQ_r'(0) = 0. \quad (2)\end{aligned}$$

If $\lambda = \mu$ then taking $\beta(v) = 0$ and $\alpha(v) = \lambda$, for any $v \in [0, 1]$, we have that (1) holds and thus $S_l(u, v)$ and $S_r(u, v)$ are C^1G^2 continuous along their common edges $S_l(1, v) = S_r(0, v)$, for all $v \in [0, 1]$.

If $P_r(0) - Q_r(0)$, $P_r'(0)$ and $Q_r'(0)$ are coplanar then there exists γ, δ and ξ such that

$$\gamma(P_r(0) - Q_r(0)) + \delta P_r'(0) + \xi Q_r'(0) = 0,$$

with $\gamma \neq 0$ or $\delta \neq 0$ or $\xi \neq 0$.

Hence, taking $\beta(v) = \gamma$, for any $v \in [0, 1]$ and

$$\alpha(v) = \begin{cases} \dfrac{\delta v + \xi(1 - v)}{2v(1 - v)} - (\lambda - \mu) & \text{if } 0 < v < 1, \\ \lambda - \delta & \text{if } v = 0, \\ \mu - \gamma & \text{if } v = 1, \end{cases}$$

then from (2) we conclude that (1) holds and thus $S_l(u, v)$ and $S_r(u, v)$ are C^1G^2 continuous along their common edges $S_l(1, v) = S_r(0, v)$, for all $v \in [0, 1]$. □

3. Linear interpolation of C^1G^2 continuous surfaces

Let us consider the blending interpolation between two pairs of surfaces $P_l(u, v)$, $P_r(u, v)$ and $Q_l(u, v)$, $Q_r(u, v)$, with $u, v \in [0, 1]$. We assume that $P_l(u, v)$, $P_r(u, v)$ are C^1G^2 continuous along edges $P_l(1, v) = P_r(0, v)$, with $v \in [0, 1]$, that implies that there exist $\alpha_p(v)$ and $\beta_p(v)$, with $v \in [0, 1]$, such that

$$\frac{\partial P_l}{\partial u}(1, v) = \frac{\partial P_r}{\partial u}(0, v),$$
$$\frac{\partial^2 P_l}{\partial u^2}(1, v) = \frac{\partial^2 P_r}{\partial u^2} + \alpha_p(v)\frac{\partial P_r}{\partial u}(0, v) + \beta_p(v)\frac{\partial P_r}{\partial v}(0, v).$$

Moreover, we suppose that $Q_l(u, v)$, $Q_r(u, v)$ are C^1G^2 continuous along edges $Q_l(1, v) = Q_r(0, v)$, with $v \in [0, 1]$, that implies that there exist $\alpha_q(v)$ and $\beta_q(v)$, with $v \in [0, 1]$, such that

$$\frac{\partial Q_l}{\partial u}(1, v) = \frac{\partial Q_r}{\partial u}(0, v),$$
$$\frac{\partial^2 Q_l}{\partial u^2}(1, v) = \frac{\partial^2 Q_r}{\partial u^2} + \alpha_q(v)\frac{\partial Q_r}{\partial u}(0, v) + \beta_q(v)\frac{\partial Q_r}{\partial v}(0, v).$$

Theorem 2. *For any $\omega \in [0, 1]$, the result blending surfaces $R_l(u, v) = (1 - \omega)P_l(u, v) + \omega Q_l(u, v)$ and $R_r(u, v) = (1 - \omega)P_r(u, v) + \omega Q_r(u, v)$ may not be C^1G^2 continuous.*

Proof. Let $\omega \in [0, 1]$ and suppose that $R_l(u, v)$ and $R_r(u, v)$ are C^1G^2 continuous along their common edges $R_l(u, v) = R_r(1, v)$, for any $v \in [0, 1]$, then there exist two functions $\alpha_r(v)$ and $\beta_r(v)$, $v \in [0, 1]$ such that for any $v \in [0, 1]$,

$$\frac{\partial R_l}{\partial u}(1, v) = \frac{\partial R_r}{\partial u}(0, v),$$
$$\begin{aligned}\frac{\partial^2 R_l}{\partial u^2}(1, v) &= \frac{\partial^2 R_r}{\partial u^2}(0, v) + \alpha_r(v)\partial R_r \partial u(0, v) \\ &\quad + \beta_r(v)\frac{\partial R_r}{\partial v}(0, v).\end{aligned}$$

This implies that

$$\begin{aligned}&(1 - \omega)\frac{\partial^2 P_l}{\partial u^2}(1, v) + \omega\frac{\partial^2 Q_l}{\partial u^2}(1, v) \\ &= (1 - \omega)\frac{\partial^2 P_r}{\partial u^2}(0, v) + \omega\frac{\partial Q_r}{\partial u^2}(0, v) \\ &\quad + \alpha_r(v)\left((1 - \omega)\frac{\partial P_r}{\partial u}(0, v) + \omega\frac{\partial Q_r}{\partial u}(0, v)\right) \\ &\quad + \beta_r(v)\left((1 - \omega)\frac{\partial P_r}{\partial v}(0, v) + \omega\frac{\partial Q_r}{\partial v}(0, v)\right).\end{aligned}$$

Thus

$$\begin{aligned}&(1 - \omega)\left(\frac{\partial^2 P_l}{\partial u^2}(0, v) + \alpha_p(v)\frac{\partial P_r}{\partial u}(0, v) + \beta_p\frac{\partial P_r}{\partial v}(0, v)\right) \\ &\quad + \omega\left(\frac{\partial^2 Q_r}{\partial u^2}(0, v) + \alpha_q(v)\frac{\partial Q_r}{\partial u} + \beta_q(v)\frac{\partial Q_r}{\partial v}(0, v)\right) \\ &= (1 - \omega)\frac{\partial^2 P_r}{\partial u^2}(0, v) + \omega\frac{\partial Q_r}{\partial u^2}(0, v) \\ &\quad + \alpha_r(v)\left((1 - \omega)\frac{\partial P_r}{\partial u}(0, v) + \omega\frac{\partial Q_r}{\partial u}(0, v)\right) \\ &\quad + \beta_r(v)\left((1 - \omega)\frac{\partial P_r}{\partial v}(0, v) + \omega\frac{\partial Q_r}{\partial v}(0, v)\right).\end{aligned}$$

Hence

$$\begin{aligned}&(1 - \omega)(\alpha_p(v) - \alpha_r(v))\frac{\partial P_r}{\partial u}(0, v) \\ &\quad + (1 - \omega)(\beta_p(v) - \beta_r(v))\frac{P_r}{\partial v}(0, v) + \omega(\alpha_p(v) - \alpha_r(v))\frac{\partial Q_r}{\partial u} \\ &\quad + \omega(\beta_p(v) - \beta_r(v))\frac{\partial Q_r}{\partial v} = 0, \quad \forall v \in [0, 1],\end{aligned}$$

but normally this relation is not true. □

4. Retaining C^1G^2 continuity of blended Bézier surfaces pair

Consider two Bézier surface-pairs $P_l(u, v)$, $P_r(u, v)$ and $Q_l(u, v)$, $Q_r(u, v)$, $u, v \in [0, 1]$. Assume that each Bézier surface $P_l(u, v)$, $P_r(u, v)$, $Q_l(u, v)$ and $Q_r(u, v)$ have $n + 1$ control points along the