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Algebraic methods in approximation theory

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ABSTRACT

This survey gives an overview of several fundamental algebraic constructions which arise in the study of splines. Splines play a key role in approximation theory, geometric modeling, and numerical analysis; their properties depend on combinatorics, topology, and geometry of a simplicial or polyhedral subdivision of a region in $\mathbb{R}^k$, and are often quite subtle. We describe four algebraic techniques which are useful in the study of splines: homology, graded algebra, localization, and inverse systems. Our goal is to give a hands-on introduction to the methods, and illustrate them with concrete examples in the context of splines. We highlight progress made with these methods, such as a formula for the third coefficient of the polynomial giving the dimension of the spline space in high degree. The objects appearing here may be computed using the `spline` package of the `Macaulay2` software system.

1. Introduction

In mathematics it is often useful to approximate a function f on a region by a simpler function. A natural way to do this is to divide the region into simplices or polyhedra, and then approximate f on each simplex by a polynomial function. A C^r-differentiable piecewise polynomial function on a k-dimensional simplicial or polyhedral subdivision $\Delta \subseteq \mathbb{R}^k$ is called a *spline*. Splines are ubiquitous in geometric modeling and approximation theory, and play a key role in the finite element method for solving PDEs. There is also a great deal of beautiful mathematical structure to these problems, involving commutative and homological algebra, geometry, combinatorics and topology.

For a fixed Δ and choice of smoothness r, the set of splines where each polynomial has degree at most d is a real vector space, denoted $S^r_d(\Delta)$. The dimension of $S^r_d(\Delta)$ depends on r, d and the geometric, combinatorial, and topological properties of Δ. For many important cases, there is no explicit general formula known for this dimension. In applications, it will also be important to find a good basis, or at least a good generating set for $S^r_d(\Delta)$; in this context good typically means splines which have a small support set.

Splines seem to have first appeared in a paper of Courant (1943), who considered the C^0 case. Pioneering work by Schumaker (1979) in the planar setting established a dimension formula for all d when Δ has a unique interior vertex, as well as a lower bound for any Δ:

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Theorem 1.1. *(See Schumaker (1979).) For a simplicial complex* $\Delta \subseteq \mathbb{R}^2$

$$\dim S^r_d(\Delta) \geq \binom{d+2}{2} + \binom{d-r+1}{2} f^0_1 - \left(\binom{d+2}{2} - \binom{r+2}{2}\right) f^0_0 + \sum \sigma_i$$

where $f^0_1 = |\text{interior edges}|$, $f^0_0 = |\text{interior vertices}|$, *and* $\sigma_i = \sum_j \max\{(r+1+j(1-n(v_i))), 0\}$, *with* $n(v_i)$ *the number of distinct slopes at an interior vertex* v_i.

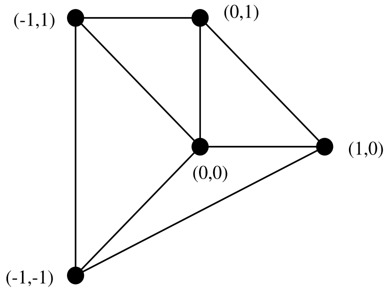
Using Bezier–Bernstein techniques, Alfeld and Schumaker (1987) prove that if $d \geq 4r+1$ then equality holds in Theorem 1.1, Hong (1991) shows equality holds if $d \geq 3r+2$, and Alfeld and Schumaker (1990) show equality holds for $d \geq 3r+1$ and Δ generic. There remain tantalizing open questions in the planar case: the Oberwolfach problem book from May 1997 contains a conjecture of Alfeld–Manni that for $r = 1$ Theorem 1.1 gives the dimension in degree $d = 3$. Works of Diener (1990) and Tohaneanu (2005) show the next conjecture is optimal.

Conjecture 1.2. *(See Schenck (1997a), Schenck and Stiller (2002).) The Schumaker formula holds with equality if* $d \geq 2r+1$.

Homological methods were introduced to the field in a watershed 1988 paper of Billera (1988), which solved a conjecture of Strang (1974) on the dimension of $S^1_2(\Delta)$ for a generic planar triangulation. One key ingredient in the work was a result of Whiteley (1991) using rigidity theory. Homological methods are discussed in detail in §2, and the utility of these tools is illustrated in §4.

A useful observation is that the smoothness condition is local: for two k simplices σ_1 and σ_2 sharing a common $k-1$ face τ, let l_τ be a nonzero linear form vanishing on τ. Then a pair of polynomials f_1, f_2 meet with order r smoothness across τ iff $l^{r+1}_\tau | f_1 - f_2$. For splines on a line, the situation is easy to understand, so the history of the subject really begins with the planar case. Even the simplest case is quite interesting: let $\Delta \subseteq \mathbb{R}^2$ be the star of a vertex, so that Δ is triangulated with a single interior vertex as in the next example.

Example 1.3. A planar Δ which is the star of a single interior vertex v_0 at the origin.



Starting with the triangle in the first quadrant and moving clockwise, label the polynomials on the triangles $f_1, \ldots, f_4$. To obtain a global C^r function, we require

$$\begin{aligned} a_1 y^{r+1} &= f_1 - f_2 \\ a_2 (x-y)^{r+1} &= f_2 - f_3 \\ a_3 (x+y)^{r+1} &= f_3 - f_4 \\ a_4 x^{r+1} &= f_4 - f_1 \end{aligned} \tag{1}$$

Summing both sides yields the equation $\sum_{i=1}^4 a_i l_i^{r+1} = 0$ (where $l_1 = y$ and so on), and gives a hint that algebra has a role to play.

Definition 1.4. Let $\{f_1, \ldots, f_m\}$ be a set of polynomials. A syzygy is a relation

$$\sum_{i=1}^m a_i f_i = 0, \text{ where the } a_i \text{ are also polynomials.}$$

Notice that if each f_i is a fixed polynomial f, then the smoothness condition is trivially satisfied.

Definition 1.5. For any r, d, Δ, the set of polynomials of degree at most d is a subspace of $S^r_d(\Delta)$, which we call global polynomials.