Computer Aided Geometric Design 45 (2016) 48–72

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Polynomial finite element method for domains enclosed by piecewise conics

Oleg Davydov [a,*], Georgy Kostin [b], Abid Saeed [c]

[a] Department of Mathematics, University of Giessen, Arndtstrasse 2, 35392 Giessen, Germany
[b] Institute for Problems in Mechanics, Russian Academy of Sciences, 101-1, Vernadskogo pr., Moscow, 117526, Russia
[c] Department of Mathematics, Kohat University of Science and Technology, Kohat, Pakistan

ARTICLE INFO

Article history:
Available online 23 November 2015

Keywords:
Multivariate splines
Curved finite elements

ABSTRACT

We consider bivariate piecewise polynomial finite element spaces for curved domains bounded by piecewise conics satisfying homogeneous boundary conditions, construct stable local bases for them using Bernstein–Bézier techniques, prove error bounds and develop optimal assembly algorithms for the finite element system matrices. Numerical experiments confirm the effectiveness of the method.

1. Introduction

Spaces of multivariate piecewise polynomial splines are usually defined on triangulated polyhedral domains without imposing any boundary conditions. However, applications such as the finite element method require at least the ability to prescribe zero values on parts of the boundary. Fitting data with curved discontinuities of the derivatives is another situation where the interpolation of prescribed values along a lower dimensional manifold is highly desirable. It turns out that such conditions make the otherwise well understood spaces of e.g. bivariate C^1 macro-elements on triangulations significantly more complex. Even in the simplest case of a polygonal domain, the dimension of the space of splines vanishing on the boundary is dependent on its geometry, with consequences for the construction of stable bases (or stable minimal determining sets) (Davydov and Saeed, 2012, 2013).

Since splines are piecewise polynomials, it is convenient to model curved features by piecewise algebraic surfaces so that the spline space naturally splits out the subspace of functions vanishing on such a surface. Indeed, implicit algebraic surfaces are a well-established modeling tool in CAGD (Bloomenthal et al., 1997), and the ability to exactly reproduce some of them (e.g. circles or cylinders) is a highly desirable feature for any modeling method (Farin, 2002).

On the other hand, the finite element analysis benefits a lot from the isogeometric approach (Hughes et al., 2005), where the geometric models of the boundary are used exactly in the form they exist in a CAD system rather than undergoing a remeshing to fit into the traditional isoparametric finite element scheme. While the isogeometric analysis introduced in Hughes et al. (2005) is based on the most widespread modeling tool of NURBS and benefits from the many attractive features of tensor-product B-splines, it also inherits some of their drawbacks, such as complicated local refinement (see for example Buffa et al., 2010).

In this paper we explore an isogeometric method which combines modeling with algebraic curves with the standard triangular piecewise polynomial finite elements in the simplest case of planar domains defined by piecewise quadratic alge-

* Corresponding author.
E-mail addresses: oleg.davydov@math.uni-giessen.de (O. Davydov), kostin@ipmnet.ru (G. Kostin), abidsaeed@kust.edu.pk (A. Saeed).

http://dx.doi.org/10.1016/j.cagd.2015.11.002
0167-8396/

braic curves (conic sections). Remarkably, the standard Bernstein–Bézier techniques for dealing with piecewise polynomials on triangulations (Lai and Schumaker, 2007; Schumaker, 2015) as well as recent optimal assembly algorithms (Ainsworth et al., 2011, 2015a, 2015b) for high order elements can be carried over to this case without significant loss of efficiency. Some of the material, especially in Sections 4 and 6 is based on the thesis (Saeed, 2012) of one of the authors. Note that we only consider C^0 elements for elliptic problems with homogeneous Dirichlet boundary conditions, although preliminary results on a direct implementation of non-homogeneous Dirichlet boundary conditions can be found in Saeed (2012).

In contrast to both the isoparametric curved finite elements and the isogeometric analysis, our approach does not require parametric patching on curved subtriangles, and therefore does not depend on the invertibility of the Jacobian matrices of the nonlinear geometry mappings. Therefore our finite elements remain piecewise polynomial everywhere in the physical domain. This in particular facilitates a relatively straightforward extension to C^1 elements on piecewise conic domains, which have also been considered in Saeed (2012) and tested numerically on the approximate solution of fully nonlinear elliptic equations by Böhmer's method (Böhmer, 2008). Full details of the theory of these elements are postponed to our forthcoming paper (Davydov and Saeed, in preparation).

There are some connections to the weighted extended B-spline (web-spline) method (Höllig et al., 2001). In particular, in our error analysis we use a technical lemma (Lemma 3.1) proved in Höllig et al. (2001). Indeed, the quadratic polynomials that define the curved edges of the pie-shaped triangles at the domain boundary are factored out from the local polynomial spaces and hence act as weight functions on certain subdomains. They remain however integral parts of the spline spaces in our case and are generated naturally from the conic sections defining the domain, thus bypassing the problem of the computation of a smooth global weight function needed in the web-spline method.

The paper is organized as follows. We introduce in Section 2 the spaces $S_{d,0}(\triangle)$ of C^0 piecewise polynomials of degree d on domains bounded by a number of conic sections, with homogeneous boundary conditions and investigate in Section 3 their approximation power for functions in Sobolev spaces $H^m(\Omega)$ vanishing on the boundary, which leads in particular to the error bounds in the form $\mathcal{O}(h^m)$ in the L_2-norm and $\mathcal{O}(h^{m-1})$ in the H^1-norm for the solutions of elliptic problems by the Ritz–Galerkin finite element method. Section 4 is devoted to the development of a basis for $S_{d,0}(\triangle)$ of Bernstein–Bézier type important for a numerically stable and efficient implementation of the method. Some implementation issues specific for the curved elements are treated in Section 5, including the fast assembly of the system matrices. Finally, Section 6 presents several numerical experiments involving the Poisson problem on two different curved domains, as well as the circular membrane eigenvalue problem. The results confirm the effectiveness of our method both in h- and p-refinement settings.

2. Piecewise polynomials on piecewise conic domains

Let $\Omega \subset \mathbb{R}^2$ be a bounded curvilinear polygonal domain with $\Gamma = \partial\Omega = \bigcup_{j=1}^n \overline{\Gamma}_j$, where each Γ_j is an open arc of an algebraic curve of at most second order (*i.e.*, either a straight line or a conic). For simplicity we assume that Ω is simply connected. Let $Z = \{z_1, \dots, z_n\}$ be the set of the endpoints of all arcs numbered counter-clockwise such that z_j, z_{j+1} are the endpoints of Γ_j, $j = 1, \dots, n$. (We set $z_{j+n} = z_j$.) Furthermore, for each j we denote by ω_j the internal angle between the tangents τ_j^+ and τ_j^- to Γ_j and Γ_{j-1}, respectively, at z_j. We assume that $0 < \omega_j \le 2\pi$, and set $\omega := \min\{\omega_j : 1 \le j \le n\}$.

Our goal is to develop an H^1-conforming finite element method with polynomial shape functions suitable for solving second order elliptic problems on curvilinear polygons of the above type.

Let $\triangle$ be a *triangulation* of Ω, *i.e.*, a subdivision of Ω into triangles, where each triangle $T \in \triangle$ has at most one edge replaced with a curved segment of the boundary $\partial\Omega$, and the intersection of any pair of the triangles is either a common vertex or a common (straight) edge if it is non-empty. The triangles with a curved edge are said to be *pie-shaped*. Any triangle $T \in \triangle$ that shares at least one edge with a pie-shaped triangle is called a *buffer* triangle, and the remaining triangles are *ordinary*. We denote by $\triangle_0$, $\triangle_B$ and $\triangle_P$ the sets of all ordinary, buffer and pie-shaped triangles of $\triangle$, respectively. Thus,

$$\triangle = \triangle_0 \cup \triangle_B \cup \triangle_P$$

is a disjoint union, see Fig. 1. We emphasize that a triangle with only straight edges on the boundary of Ω does not belong to $\triangle_P$.

We denote by $\mathbb{P}_d$ the space of all bivariate polynomials of total degree at most d. For each $j = 1, \dots, n$, let $q_j \in \mathbb{P}_2$ be a polynomial such that $\Gamma_j \subset \{x \in \mathbb{R}^2 : q_j(x) = 0\}$. By multiplying q_j by -1 if needed, we ensure that $\partial_{\nu_x} q_j(x) < 0$ for all x in the interior of Γ_j, where ν_x denotes the unit outer normal to the boundary at x, and $\partial_a := a \cdot \nabla$ is the directional derivative with respect to a vector a. Hence, $q_j(x)$ is positive for points in Ω near the boundary segment Γ_j. We assume that $q_j \in \mathbb{P}_1$ or $q_j \in \mathbb{P}_2 \setminus \mathbb{P}_1$ depending on whether Γ_j is a straight interval or a genuine conic arc.

Furthermore, let V, E, V_I, E_I, V_B and E_B denote the set of all vertices, all edges, interior vertices, interior edges, boundary vertices and boundary edges of $\triangle$, respectively. For each $v \in V$, $\mathrm{star}(v)$ stands for the union of all triangles in $\triangle$ attached to v. We also denote by θ the smallest angle of the triangles $T \in \triangle$, where the angle between an interior edge and a boundary segment is understood in the tangential sense.