Computer Aided Geometric Design 45 (2016) 73–82

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Interpolation properties of C^1 quadratic splines on hexagonal cells

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ARTICLE INFO

Article history:
Available online 7 January 2016

Keywords:
Spline
Unconfinable vertex
Interpolation
Projective geometry

ABSTRACT

Let Δ_n be a cell with a single interior vertex and n boundary vertices $v_1, \ldots, v_n$. Say that Δ_n has the interpolation property if for every $z_1, \ldots, z_n \in \mathbb{R}$ there is a spline $s \in \mathcal{S}_2^1(\Delta_n)$ such that $s(v_i) = z_i$ for all i. We investigate under what conditions does a cell fail the interpolation property. The question is related to an open problem posed by Alfeld, Piper, and Schumaker in 1987 about characterization of unconfinable vertices.

For hexagonal cells, we obtain a geometric criterion characterizing the failure of the interpolation property. As a corollary, we conclude that a hexagonal cell such that its six interior edges lie on three lines fails the interpolation property if and only if the cell is projectively equivalent to a regular hexagonal cell. Along the way, we obtain an explicit basis for the vector space $\mathcal{S}_2^1(\Delta_n)$ for $n \geq 5$.

0. Introduction

Suppose that A is a finite set of points in $\mathbb{R}^2$. The condition that a polynomial of a fixed degree vanishes on A can be expressed as a system of linear equations in the coefficients of the polynomial. A classically studied problem asks under what conditions on A are these equations linearly independent. One example of a (classical) answer to that question appears in Fact 2.3. We consider a version of this problem, replacing polynomial functions with splines.

Definition 0.1. Let Δ_6 be a triangulation with one interior vertex v^* and six boundary vertices $v_1, \ldots, v_6$. We say that Δ_6 *has the interpolation property* if for every $z_1, \ldots, z_6 \in \mathbb{R}$ there is $s \in \mathcal{S}_2^1(\Delta_6)$ such that $s(v_i) = z_i$ for $i = 1, \ldots, 6$.

We want to know when a cell Δ_6 has the interpolation property. This problem is related to the one posed by Alfeld, Piper, and Schumaker in Alfeld et al. (1987). They ask under what conditions do all the vertices of a cell Δ_n, including the interior vertex, fail to impose independent conditions on $\mathcal{S}_2^1(\Delta_n)$ (in the terminology of Alfeld et al. (1987), the interior vertex of Δ_n is *unconfinable*). It is clear that if Δ_n fails the interpolation property, then the interior vertex of Δ_n is unconfinable; but we do not know whether the converse is true.

It was shown in Alfeld et al. (1987) that if the interior vertex of Δ_n is unconfinable, then $n \geq 6$ and n is even. For $n = 6$, Alfeld et al. (1987) show that if Δ_6 is the triangulation of a regular (up to an affine transformation) hexagon such that its 6 interior edges lie on 3 lines, then Δ_6 fails the interpolation property (and thus is unconfinable). But it was not clear,

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http://dx.doi.org/10.1016/j.cagd.2016.01.001
0167-8396/

for instance, whether this is the only example of a hexagonal cell with an unconfinable interior vertex. The problem was mentioned again in Alfeld (2000).

We show in Corollary 2.8 that if Δ_6 is a triangulation such that all of its interior edges lie on three lines, then Δ_6 fails the interpolation property if and only if the hexagon is regular up to a *projective* transformation. We obtain the corollary as a consequence of a more general characterization in Theorem 2.2. Both the corollary and the theorem allow us to describe new classes of hexagonal cells with an unconfinable interior vertex. It is worth noting that the characterization of the interpolation property we find has a distinct geometric flavor; this was also the case in the study of interpolation properties of linear splines (see Davydov et al., 2000, and references therein).

A key tool in the analysis is a convenient explicit basis for the vector space $\mathcal{S}_2^1(\Delta_n)$, $n \geq 5$, obtained in Proposition 1.1. This is the content of Section 1. The rest of the paper is organized as follows. Section 2 contains the statements of the main results: Theorem 2.2 and Corollary 2.8. The proof of Theorem 2.2 is also in Section 2, modulo the proofs of two lemmas. We chose to separate the proofs of the two technical lemmas to Section 3 to make the structure of the main argument more transparent. Corollary 2.8 is proved in Section 4.

1. Explicit basis for C^1 quadratic splines on a cell

Let Δ_n be a cell with the interior vertex $v^* = (0, 0)$. Given a counterclockwise sequential labeling $\{v_i \mid 1 \leq i \leq n\}$ of the boundary vertices of Δ_n, we will denote, for $i = 1, \ldots, n$, by τ_i the edge containing v^* and v_i and by σ_i the triangle with vertices v^*, v_i, and v_{i+1}. We will use the convention that the index arithmetic is the arithmetic modulo n (so, for example, $v_n = v_0$, $v_{n+1} = v_1$, and so on).

Let ℓ_i be a linear form such that the line $\ell_i(x, y) = 0$ contains the edge τ_i. We will take the gradient of ℓ_i to be the unit vector in the direction $\langle -y_i, x_i \rangle$, where (x_i, y_i) are the coordinates of v_i. Let $\theta_{i,j}$, $1 \leq i < j \leq n$ denote the angle between the edges τ_i and τ_j (or, equivalently, the angle between the gradients of ℓ_i and ℓ_j).

The symbol $d(v_j, \ell_i)$ will denote the (oriented) distance from the vertex v_j to the line $\ell_i = 0$. Given the convention that the gradient of ℓ_i is a unit vector, $d(v_j, \ell_i)$ is simply the value $\ell_i(v_j)$; we use the notation $d(v_j, \ell_i)$ to highlight that we are thinking of the quantity as the distance.

The following proposition is the main result of this section.

Proposition 1.1. *Let Δ_n be a cell, $n \geq 5$. There is a sequential counterclockwise labeling of the boundary vertices v_i, $i = 1, \ldots, n$, and a family $\{s_i \mid s_i \in \mathcal{S}_2^1(\Delta_n),\ i = 1, \ldots, n-3\}$ such that*

1. *for each $i = 1, \ldots, n-3$, the support of s_i is contained in the set $\sigma_i \cup \sigma_{i+1} \cup \sigma_{i+2}$ and $s_i(v_{i+1}) \cdot s_i(v_{i+2}) \neq 0$;*
2. *for each $i = 1, \ldots, n-3$, we have*
$$\frac{s_i(v_{i+1})}{s_i(v_{i+2})} = \frac{d(v_{i+1}, \ell_i) \cdot d(v_{i+1}, \ell_{i+3})}{d(v_{i+2}, \ell_i) \cdot d(v_{i+2}, \ell_{i+3})};$$
3. *the set of functions $\mathcal{B} = \{1, x, y, x^2, xy, y^2, s_1, \ldots, s_{n-3}\}$ is a basis for $\mathcal{S}_2^1(\Delta_n)$.*

Definition 1.2. A spline s_i that satisfies (1) and (2) of Proposition 1.1 will be called a *basic spline*.

The proof will consist of two parts: we first establish the existence of a single basic spline under a technical assumption about the angles between certain edges (Lemma 1.4). We then show in Lemma 1.5 that one can always label the boundary vertices so that the technical assumption is satisfied for sufficiently many edges.

For Lemma 1.4, we will need the following identity.

Fact 1.3. *Let $a_i, b_i \in \mathbb{R}$ for $i \in \{1, 2, 3\}$. Then*

$$\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_1 b_1 & a_2 b_2 & a_3 b_3 \\ b_1^2 & b_2^2 & b_3^2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}.$$

The equality can be either verified directly, or obtained using Vandermonde determinant formula, so we omit the proof.

Lemma 1.4. *Fix $i \in \{1, 2, \ldots, n-3\}$. Suppose that the angles $\theta_{i,i+2}$ and $\theta_{i+1,i+3}$ are both acute. Then:*

(I) *There is a non-zero vector $(k_{i,0}, \ldots, k_{i,3})$, unique up to a constant multiple, such that for all $(x, y) \in \mathbb{R}^2$*

$$\sum_{j=0}^{3} k_{i,j} \ell_{i+j}^2(x, y) = 0. \qquad (1)$$