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On recursive refinement of convex polygons

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ABSTRACT

It is known that one can improve the accuracy of the finite element solution of partial differential equations (PDE) by uniformly refining a triangulation. Similarly, one can uniformly refine a quadrangulation. Recently polygonal meshes have been used for numerical solution of partial differential equations based on virtual element methods, weak Galerkin methods, and polygonal spline methods. A refinement scheme of pentagonal partition was introduced in Floater and Lai (2016). It is natural to ask if one can create a hexagonal refinement or general polygonal refinement schemes. In this short article, we show that one cannot refine a convex hexagon using convex hexagons of smaller size. In general, we show that one can only refine a convex n-gon by convex n-gons of smaller size if $n \leq 5$.

1. Introduction

In the fields of computer aided geometry design of surfaces and numerical solutions of partial differential equations (PDE), triangulations have been the traditional way of partitioning spatial domains. Due to the recent development of the virtual element methods, weak Galerkin methods, and polygonal splines (see Beirao da Veiga et al., 2011, 2013; Manzini et al., 2014; Rand et al., 2014; Wang and Wang, 2014; Floater and Lai, 2016), one is able to use an arbitrary polygonal partition for numerical solutions of PDE. In addition, generalized barycentric coordinates (GBC) over arbitrary polygons of n sides, n-gon for short, were invented for surface applications. See a recent survey in Floater (2015). An excellent polygonal mesh generator can be found in Talischi et al. (2012). It is known that we can uniformly refine a triangulation and a quadrangulation (cf. Lai and Schumaker, 2007) which is a common strategy to demonstrate the accuracy as well as the convergence of a numerical algorithm for solving a PDE. Recall the standard theory of spline approximation (cf. e.g. Lai and Schumaker, 2007) and the finite element method (cf. e.g. Brenner and Scott, 1994), i.e., the h-version and hp-version of finite element method requires the size of a underlying partition go to zero. It is important to have a scheme to generate partitions with finer sizes. Refining an existing partition to a partition of the same type with smaller size is an obvious approach which can be conveniently applied repeatedly to reduce the size of underlying partition. In addition, for polynomial finite elements or bivariate splines (cf. Awanou et al., 2005), the uniform refinement of triangulations/quadrangulations enables the spline spaces to have the nestedness property of the function spaces which can be important for several applications, e.g. construction of a multi-resolution analysis which leads to wavelets or tight wavelet frames (cf. e.g. Guo and Lai, 2013) as well as construction of multi-grid methods for numerical solutions of PDE (cf. e.g. Brenner and Scott, 1994). Another

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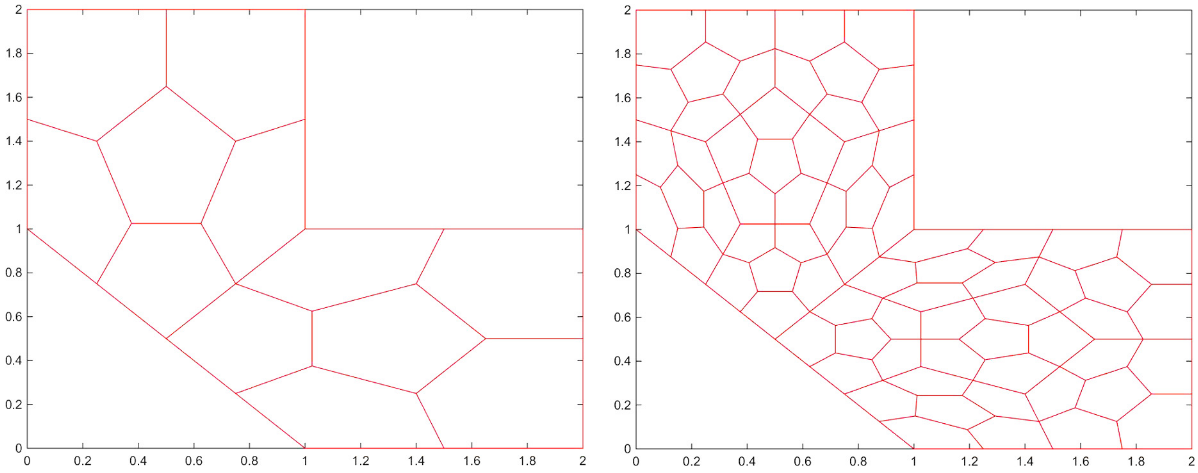

Fig. 1. A pentagonal partition (left) and its refinement (right).

important feature of uniformly refining an underlying partition is to make a computer code easy to implement and efficient to run.

Recently a refinement scheme of pentagonal partitions was introduced in Floater and Lai (2016), pictured in Fig. 1, and used to reduce the error in numerical solutions based on polygonal splines which consist of generalized Bernstein–Bézier functions in terms of GBC.

A natural question to ask is if one can create a hexagonal refinement, i.e. refine a convex hexagon by using convex hexagons of smaller size. In general, one can ask if one can create a general polygonal refinement scheme. In this short article, we will show that one cannot refine a convex hexagon by convex hexagons only. In fact, our arguments prove more. That is, one cannot refine a convex n-gon by convex n-gons of smaller size whenever $n \geq 6$. Hence, if one uses a polygonal mesh of single polygon type, then one cannot expect to generate the mesh starting from a few seeded convex n-gons with $n \geq 6$ by a recursive refinement scheme. This result will be shown in the next section. Then we shall discuss how to refine a general n-gon. We introduce a simple remedy refinement scheme of hexagons by using pentagons and one hexagon of smaller size. Similarly, a general convex n-gon can be refined by using pentagons and a convex n-gon of smaller size. In addition, we shall pose a few open questions about the possibility of refining a domain of general shape by using pentagons only. All these will be contained in §3.

2. Main results and proofs

2.1. Partitions of polygons

Definition 2.1. Let $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^2$ be a set of points. An edge e_k connecting v_{i_k} to v_{j_k} for some i_k and j_k in $\{1, 2, \ldots, n\}$ is defined as $e_k = \{x \in \mathbb{R}^2 \mid x = tv_{i_k} + (1-t)v_{j_k}, 0 \leq t \leq 1\}$. Let $E = \{e_k\}_{k=1}^{n}$ be a set of edges. We say $P = (V, E)$ is a polygon with vertices V and edges E if

(1) $\forall v \in V$, there exists exactly two distinct edges $e_{k_1}, e_{k_2} \in E$ such that $e_{k_1} \cap e_{k_2} = v$;
(2) $\forall e_{k_1}, e_{k_2}$, distinct, $e_{k_1} \cap e_{k_2}$ is either the empty set or exactly one vertex $v \in V$;
(3) The union of the edges in E forms a Jordan curve. The interior of the Jordan curve is called a face F of P.

The somewhat technical definition is meant to eliminate "poorly" behaved polygons which self-intersect. With this definition, polygons serve to separate $\mathbb{R}^2$ into a clear interior piece and an exterior piece.

Definition 2.2. The polygon $P = (V, E)$ is degenerate if it contains a vertex v whose two incident edges $e_{k_1}, e_{k_2} \in E$ with $v = e_{k_1} \cap e_{k_2}$ have the same slope.

The remainder of this paper will require that P is nondegenerate. Any degenerate polygon can be made nondegenerate by simply fusing the two edges which has the same slope (including the slope of infinity) into a single edge and omitting the vertex where they intersect.

Definition 2.3. A partition of a polygon $P = (V, E)$ is a planar graph $\hat{\mathcal{G}} = \{\hat{V}, \hat{E}, \hat{F}\}$ with vertices $\hat{V}$, edges $\hat{E}$ and faces $\hat{F}$ such that

(1) $V \subset \hat{V}$