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Completeness of generating systems for quadratic splines on adaptively refined criss-cross triangulations

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ABSTRACT

Hierarchical generating systems that are derived from Zwart–Powell (ZP) elements can be used to generate quadratic splines on adaptively refined criss-cross triangulations. We propose two extensions of these hierarchical generating systems, firstly decoupling the hierarchical ZP elements, and secondly enriching the system by including auxiliary functions. These extensions allow us to generate the *entire* hierarchical spline space – which consists of all piecewise quadratic $\mathcal{C}^1$-smooth functions on an adaptively refined criss-cross triangulation – if the triangulation fulfills certain technical assumptions. Special attention is dedicated to the characterization of the linear dependencies that are present in the resulting enriched decoupled hierarchical generating system.

1. Introduction

We consider spline spaces spanned by Zwart–Powell (ZP) elements, which were originally introduced by Zwart (1973). These functions are compactly supported $\mathcal{C}^1$-smooth piecewise quadratic spline functions defined on criss-cross triangulations, which are also called four-directional grids or type-2 triangulations (Wang, 2001). ZP elements were also recognized as a special instance of box splines, see de Boor et al. (1993) and the references cited therein.

The approximation power of ZP elements has been studied thoroughly in the literature. It was shown by Lyche et al. (2008) that no local quasi-interpolant projector exists for the box spline space defined by translates of the ZP element. Quasi-interpolation operators and the approximation power of ZP elements have been studied by Dahmen and Micchelli (1984), Dagnino and Lamberti (2001) and Foucher and Sablonnière (2008).

Various applications of ZP elements and of the spline spaces generated by them have been described in the literature. These include computer tomography (Entezari et al., 2012; Richter, 1998), computation of isophotes (Aigner et al., 2009), approximation of medial surface transforms (Bastl et al., 2010), generation of offset surfaces (Bastl et al., 2008) and numerical simulation (Kang et al., 2014).

The underlying criss-cross triangulation, which is associated with the ZP elements, possesses a highly regular structure, which precludes the possibility of adaptive refinement. This is similar to the case of tensor-product splines.

Hierarchical splines are one of the main approaches – besides T-splines (Sederberg et al., 2003), PHT-splines (Deng et al., 2008) and LR splines (Dokken et al., 2013) – that were developed to overcome this limitation. Their construction was originally proposed by Forsey and Bartels (1988). About ten years later, Kraft (1997) published a selection mechanism that defines a basis for hierarchial splines and designed a quasi-interpolation operator. Yvart et al. (2005) studied hierarchical

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triangular splines, whereas Speleers et al. (2009) explored hierarchies of Powell–Sabin splines. Recently, Giannelli et al. (2012) proposed truncated hierarchical B-splines as a modification of Kraft's basis for hierarchical tensor-product splines to restore the partition of unity property without scaling and to improve numerical stability and sparsity properties. Quasi-interpolation operators in this framework were discussed by Speleers and Manni (2016).

The applications reported in the literature include surface fitting and reconstruction (Greiner and Hormann, 1997; Kiss et al., 2014) and isogeometric analysis (Vuong et al., 2011; Schillinger et al., 2012). The idea of truncated hierarchical B-splines has recently been generalized to spaces of functions spanned by generating systems that are possibly linearly dependent (Zore and Jüttler, 2014; Kang et al., 2014). In particular, these studies include the case of ZP elements.

In fact, Kang et al. (2014) in their paper use adaptively refined spaces spanned by hierarchical ZP elements for isogeometric analysis. Their result concerning linear independence (Lemma 4) unfortunately contradicts the related result of Zore and Jüttler (2014) and relies on local linear independence, which is not satisfied by ZP elements, not even after removing one of them to restore linear independence.

The completeness problems for spline spaces is one of the fundamental questions in spline theory. Given a partition of the domain into cells (e.g., a triangulation) and a generating system of smooth piecewise polynomial functions defined on it, does it span the entire spline space consisting of the piecewise polynomial functions with a specified order of smoothness? It is closely related to the computation of the dimension of a spline space. See the monographs of Lai and Schumaker (2007) and Wang (2001) for a detailed introduction.

Several approaches to answer the completeness question and to compute the dimension have been proposed in the literature. The very powerful approach of using techniques from homological algebra was introduced by Billera (1988) and has been further explored since, e.g., by Mourrain (2014) and Schenck and Stillman (1997). The important particular case of biquadratic splines on hierarchical T-meshes has been extensively studied using the smoothing cofactor method (e.g., Huang et al., 2006; Zeng et al., 2015), and similar results are available for higher degrees as well. Indeed, the case of tensor-product splines of general degree on hierarchical meshes has attracted particular attention due to the importance for applications.

Homological techniques for hierarchical tensor-product splines were used by Berdinsky et al. (2014, 2015). The characterization of the contact of polynomials via blossoming led to the results by Mokriš et al. (2014) and Mokriš and Jüttler (2014). These generalize the earlier results of Giannelli and Jüttler (2013) on bivariate splines to the full multivariate case.

The present paper extends the latter approach to the case of bivariate splines on adaptively refined criss-cross triangulations, which are obtained by collecting triangles from a hierarchy of nested criss-cross triangulations. The case of triangulations with only one level is studied in Section 2. Considering triangles from criss-cross triangulations of several levels simultaneously leads us to the definition of the multilevel spline space, which is formalized in Section 3.

Since ZP elements are defined on each of the criss-cross triangulations that form the hierarchy, it is natural to produce a generating system by selecting ZP elements of the different levels using a suitably modified version of Kraft's construction. In order to achieve completeness of this generating system, and to control its linear dependencies, we propose to decouple the ZP elements and to enrich the hierarchical generating system by additional functions, see Section 4. Suitably generalizing the techniques of Giannelli and Jüttler (2013) then enables us to establish the completeness property under certain technical assumptions in Section 5 and to analyze the resulting linear dependencies in Section 6.

2. Zwart–Powell elements on multicell domains

After reviewing some well-known facts about Zwart–Powell box splines, we show that their restrictions span the entire space of $\mathcal{C}^1$-smooth quadratic splines on a multicell domain.

We consider the points lying on horizontal, vertical and diagonal lines

$$x = 2^{-\ell} i \text{ or } y = 2^{-\ell} i \text{ or } x + y = 2^{-\ell} i \text{ or } x - y = 2^{-\ell} i, \quad i \in \mathbb{Z},$$

which form a four-directional grid $\mathcal{G}^\ell$ in the plane $\mathbb{R}^2$. Varying the integer ℓ, which is called the *level*, leads to dyadic refinement and coarsening of the grid. While we keep it constant in this section (choosing, e.g., $\ell = 1$), we shall consider grids of different levels in the later part of the paper.

The closures of the connected components of $\mathbb{R}^2 \setminus \mathcal{G}^\ell$ are called the *cells*, and the set formed by them will be denoted by C^ℓ. Any subset thereof is called a *criss-cross triangulation* of level ℓ, see Fig. 1. More precisely, we shall speak of a criss-cross triangulation of the domain that is covered by these triangles; often we will omit the information about the level if it is clear from the context.

More generally, any set Δ of triangles in $\mathbb{R}^2$ with mutually disjoint interiors is called a *triangulation*. The *union operator* $U(\Delta)$, which is defined by

$$U(\Delta) = \bigcup_{c \in \Delta} c,$$

transforms each triangulation into a planar domain covered by its triangles.

If $\Delta^\ell \subset C^\ell$ is a criss-cross triangulation of level ℓ, then the set $D^\ell = U(\Delta^\ell) \subset \mathbb{R}^2$ shall be called a *multicell domain of level* ℓ. Clearly it is then also a multicell domain of any level ℓ' larger than ℓ, as we consider nested grids.