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Dimension and bases for geometrically continuous splines on surfaces of arbitrary topology

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ABSTRACT

We analyze the space of geometrically continuous piecewise polynomial functions, or splines, for rectangular and triangular patches with arbitrary topology and general rational transition maps. To define these spaces of G^1 spline functions, we introduce the concept of topological surface with gluing data attached to the edges shared by faces. The framework does not require manifold constructions and is general enough to allow non-orientable surfaces. We describe compatibility conditions on the transition maps so that the space of differentiable functions is ample and show that these conditions are necessary and sufficient to construct ample spline spaces. We determine the dimension of the space of G^1 spline functions which are of degree $\leqslant k$ on triangular pieces and of bi-degree $\leqslant (k,k)$ on rectangular pieces, for k big enough. A separability property on the edges is involved to obtain the dimension formula. An explicit construction of basis functions attached respectively to vertices, edges and faces is proposed; examples of bases of G^1 splines of small degree for topological surfaces with boundary and without boundary are detailed.

1. Introduction

The accurate and efficient representation of shapes is a major challenge in geometric modeling. To achieve high order accuracy in the representation of curves, surfaces or functions, piecewise polynomials models are usually employed. Parametric models with prescribed regularity properties are nowadays commonly used in Computer Aided Geometric Design (CAGD) to address these problems. They involve so-called spline functions, which are piecewise polynomial functions on intervals of $\mathbb{R}$ with continuity and differentiability constraints at some nodes. Extensions of these functions to higher dimension are usually done by taking tensor product spline basis functions. Curves, surfaces or volumes are represented as the image of parametric functions expressed in terms of spline basis functions. For instance, surface patches are described as the image of a piecewise polynomial (or rational) map from a rectangular domain of $\mathbb{R}^2$ to $\mathbb{R}^n$. But to represent objects with complex topology, such maps on rectangular parameter domains are not sufficient. One solution which is commonly used in Computer-Aided Design (CAD) is to trim the B-spline rectangular patches and to “stitch” together the trimmed pieces to create the complete shape representation. This results in complex models, which are not simple to use and to modify, since structural rigidity conditions cannot easily be imposed along the trimming curve between two trimmed patches.

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To allow flexibility in the representation of shapes with complex topology, another technique called geometric continuity has been studied. Rectangular parametric surface patches are glued along their common boundary, with continuity constraints on the tangent planes (or on higher osculating spaces). In this way, smooth surfaces can be generated from quadrilateral meshes by gluing several simple parametric surfaces, forming surfaces with the expected smoothness property along the edges.

This approach builds on the theory on differential manifolds, in works such as DeRose (1985), Hahn (1989), Gregory (1989). The idea of using transition maps or reparameterizations in connection with building smooth surfaces had been used for instance by DeRose (1985) in CAGD, who gave one of the first general definitions of splines based on fixing a parametrization.

Since these initial developments, several works focused on the construction of such G^1 surfaces (Peters, 1994; Loop, 1994; Reif, 1995; Prautzsch, 1997; Catmull and Clark, 1998; Ying and Zorin, 2004; Gu et al., 2005, 2008; He et al., 2006; Della Vecchia et al., 2008; Tosun and Zorin, 2011; Bonneau and Hahmann, 2014), with polynomial, piecewise polynomial, rational or special functions and on their use in geometric modeling applications such as surface fitting or surface reconstruction (Eck and Hoppe, 1996; Shi et al., 2004; Lin et al., 2007).

The problem of investigating the minimal degree of polynomial pieces has also been considered (Peters, 2002a). Other research investigates the construction of adapted rational transition maps for a given topological structure (Beccari et al., 2014). We refer to Peters (2002b) for a review of these constructions. Constraints that the transition maps must satisfy in order to define regular spline spaces have also been identified (Peters and Fan, 2010). But it has not yet been proved that these constraints are sufficient for the constructions.

The use of G^k spline functions to approximate functions over computational domains with arbitrary topology received recently a new attention for applications in isogeometric analysis. In this context, describing the space of functions, its dimension and adapted bases is of particular importance. A family of bi-cubic spline functions was recently introduced by Wu et al. (2014) for isogeometric applications, where constant transition maps are used, which induce singular spline basis functions at extraordinary vertices. Multi-patch representations of computational domains are also used in Buchegger et al. (2015), with constant transition maps at the shared edges of rectangular faces, using an identification of Locally Refined spline basis functions. In Kapl et al. (2014), G^k continuous splines are described and the G^1 condition is transformed into a linear system of relations between the control coefficients. The case of two rectangular patches, which share an edge is analyzed experimentally. In Bercovier and Matskewich (2015), the space of G^1 splines of bi-degree $\geqslant 4$ for rectangular decompositions of planar domains is analyzed. Minimal Determining Sets of points are studied, providing dimension formulae and dual basis for G^1 spline functions over planar rectangular meshes with linear gluing transition maps.

Our objective is to analyze the space of G^1 spline functions for rectangular and triangular patches with arbitrary topology and general rational transition maps. We are interested in determining the dimension of the space of G^1 spline functions which are of degree $\leqslant k$ on triangular pieces and of bi-degree $\leqslant (k,k)$ on rectangular pieces. To define the space of G^1 spline functions, we introduce the concept of topological surface with gluing data attached to the edges shared by the faces. The framework does not require manifold constructions and is general enough to allow non-orientable surfaces. We describe compatibility conditions on the transition maps so that the space of differentiable functions is ample and show that these conditions are necessary and sufficient to construct ample spline spaces. A separability property is involved to obtain a dimension formula of the G^1 spline spaces of degree $\leqslant k$ on such topological surfaces, for k big enough. This leads to an explicit construction of basis functions attached respectively to vertices, edges and faces.

For the presentation of these results, we structure the paper as follows. The next section introduces the notion of topological surface $\mathcal{M}$, differentiable functions on $\mathcal{M}$ and constraints on the transition maps to have an ample space of differentiable functions. Section 3 deals with the space of spline functions which are piecewise polynomial and differentiable on $\mathcal{M}$. Section 4 analyzes the gluing conditions along an edge. Section 5 analyzes the gluing condition around a vertex. In Section 6, we give the dimension formula for the space of spline functions of degree $\leqslant k$ over a topological surface $\mathcal{M}$ and describe explicit basis constructions. Finally, in Section 7, we detail an example with boundary edges and another one with no boundary edges. We also provide an appendix with an algorithmic description of the basis construction.

2. Differentiable functions on a topological surface

Typically in CAGD, parametric patches are glued into surfaces by splines (i.e., polynomial maps) from polygons in $\mathbb{R}^2$. The simplest C^r construction is with the polygons in $\mathbb{R}^2$ situated next to each other, so that C^r continuity across patch edges comes from C^r continuity of the coordinate functions across the polygon edges. This is called *parametric continuity*. A more general construction to generate a C^r surface from polygonal patches is called *geometric continuity* (DeRose, 1985; Peters, 2002b). Inspired by differential geometry, attempts have been made (Hahn, 1989; Vidunas, 1999, 2003) to define *geometrically continuous G^r surfaces* from a collection of polygons in $\mathbb{R}^2$ with additional data to glue their edges and differentiations. They are defined by parametrization maps from the polygons to $\mathbb{R}^3$ satisfying geometric regularity conditions along edges.

It is easy to define a C^0 surface from a collection of polygons and homeomorphisms between their edges.