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Dimension of trivariate C^1 splines on bipyramid cells

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ABSTRACT

We study the dimension of trivariate C^1 splines on bipyramid cells, that is, cells with $n+2$ boundary vertices, n of which are coplanar with the interior vertex. We improve the earlier lower bound on the dimension given by J. Shan. Moreover, we derive a new upper bound that is equal to the known lower bound in most cases. In the remaining cases, our upper bound is close to the known lower bound, and we conjecture that the dimension coincides with the upper bound. We use tools from both algebraic geometry and Bernstein–Bézier analysis.

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1. Introduction

Smooth piecewise polynomial functions, or splines, are used widely in approximation theory, computer aided geometric design, image analysis, and numerical analysis. Key tools for analyzing multivariate spline spaces come from two fields: classical approximation theory and applied algebraic geometry. The two fields and the two corresponding research communities use entirely different approaches: Bernstein–Bézier analysis (see the survey of Alfeld, 2016, in this issue) and homological techniques (see the survey of Schenck, 2016, in this issue). For a full treatise on Bernstein–Bézier analysis, we refer the reader to Lai and Schumaker (2007). The homological approach to multivariate splines is explained in several papers, see Mourrain and Villamizar (2014), Schenck and Stilman (1997) and Billera (1988).

In this paper we are concerned with trivariate splines. Let $\mathcal{P}_d$ denote the set of polynomials in three variables of degree $\le d$. A **spline** is a piecewise-polynomial function defined on a domain $\Omega \subset \mathbb{R}^3$ that belongs to a certain smoothness class. More precisely, for a fixed tetrahedral partition $\triangle$ of the underlying domain Ω, our spline space of interest is defined as

$$S_d^r(\triangle) = \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d \quad \text{for each tetrahedron} \quad T \in \triangle\}. \tag{1}$$

Finding the exact dimensions of spaces $S_d^r(\triangle)$ of trivariate splines remains an open problem. While the dimension of any particular spline can be computed using software such as Macaulay2, or the java applet (Alfeld), general formulae are still unknown. This problem is inherited from bivariate splines, where the dimension remains unknown for $d \le 3r+1$. As it is shown in section 17.8 in Lai and Schumaker (2007), there is a tight connection between trivariate and bivariate cases, and there is no hope to resolve the trivariate case before the dimension of bivariate splines is known for all values of d and r. We note that despite this difficulty, the dimensions of trivariate splines are known for few very special partitions, including the so-called macro-element or finite-element spaces. For example, trivariate splines on the Alfeld, the Clough–Tocher and

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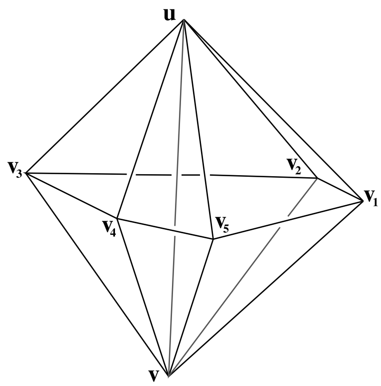

Fig. 1. Orange in $\mathbb{R}^3$.

Worsey–Piper splits, see Lai and Schumaker (2007) and references therein. Other examples do not qualify as finite-element spaces, but are built on very special grids, see Schumaker and Sorokina (2004) for an example. The first and only formula for a dimension of a trivariate spline space on a tetrahedral partition of a somewhat general nature is for the so-called oranges, see Theorem 17.30 in Lai and Schumaker (2007). An orange, see Fig. 1, is a union of tetrahedra all sharing the same edge, $[u, v]$ in Fig. 1, and pairwise sharing common triangular faces. The goal of this paper is to provide a second example of such a general formula for a partition that we call a *bipyramid cell*, see Fig. 2. A bipyramid cell is a special case of the so-called interior cell. An interior cell is a collection of tetrahedra all sharing an interior vertex (cf. Section 17.8.1 in Lai and Schumaker, 2007). Note that a bipyramid cell is very different from an orange. An orange is not an interior cell, and all smoothness conditions associated with it are essentially bivariate. The smoothness conditions associated with the bipyramid cell include purely trivariate ones. Because of this difficulty, we have been able to analyze the case for $r = 1$ in (1) only.

A large part of research in the area of trivariate splines has focused on obtaining upper and lower bounds on the dimension of trivariate spline spaces. Several results have been obtained for bounding the dimension of general splines either below or above, see Mourrain and Villamizar (2014), Lau (2005), Alfeld and Schumaker (2008), Alfeld (1996), and Shan (2014). None of the bounds cited above are tight enough to obtain exact dimensions. In this paper, we prove new upper bounds for the bipyramid cells that allow us to obtain exact dimensions in three out of four subcases; in the last subcase our new upper bound is very close to the known lower bound, and we conjecture that the dimension is equal to the upper bound.

Finally, we prove that the lower bound in Shan (2014) has a typo; fixing this leads to an improved lower bound. We compare the improved lower bound to our new upper bound. The lower bound is obtained using homological techniques, while the upper bound is proved using Bernstein–Bézier analysis.

The paper is organized as follows. In Section 2 we introduce bipyramid cells, and study their geometry. In Section 3 we fix the bound in Shan (2014), and derive the lower bound for the dimension of $S_d^1(\Delta)$ on bipyramid cells. In Section 4 we prove a result on supersmoothness needed to obtain our new upper bound. Section 5 contains several results on the dimension of bivariate splines that are also needed to derive the new upper bound. The main results of this paper concerning the new upper bound and the exact dimension of $S_d^1(\Delta)$ on bipyramid cells are in Section 6. We conclude with remarks and conjectures in Section 7.

2. Preliminaries

To begin, we define a class of partitions in $\mathbb{R}^3$ called *bipyramid cells*, see e.g. Fig. 2.

Definition 2.1. A *bipyramid cell* is a tetrahedral partition Δ such that:

- there is exactly one interior vertex v_0;
- n boundary vertices v_1 through v_n are coplanar along with v_0, and form a polygon surrounding v_0 in the *base plane* $B := [v_1, \dots, v_n]$;
- each vertex v_i, $i = 1, \dots, n$, is connected to v_0 by the interior edges $[v_0, v_i]$;
- two boundary vertices v_{n+1} and v_{n+2} lie outside the *base plane* B, are on opposite sides of B, and are connected to v_0 by the interior edges $[v_0, v_{n+1}]$ and $[v_0, v_{n+2}]$; and
- vertices v_{n+1} and v_{n+2} connect to the boundary vertices v_i, $i = 1, \dots, n$.

It is easy to check that these conditions force Δ to be *shellable* with no holes or cavities, see Lai and Schumaker (2007) for a discussion of the importance of these facts. For simplicity, and without loss of generality, we can assume that the interior vertex v_0 is located at the origin, and the n coplanar boundary vertices lie in the base xy-plane. For the remainder

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