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Principal curvature ridges and geometrically salient regions of parametric *B*-spline surfaces

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ABSTRACT

Ridges are characteristic curves of a surface that mark salient intrinsic features of its shape and are therefore valuable for shape matching, surface quality control, visualization and various other applications. Ridges are loci of points on a surface where one of the principal curvatures attain a critical value in its respective principal direction. We present a new algorithm for accurately extracting ridges on *B*-spline surfaces and define a new type of salient region corresponding to major ridges that characterize geometrically significant regions on surfaces. Ridges exhibit complex behavior near umbilics on a surface, and may also pass through certain turning points causing added complexity for ridge computation. We present a new numerical tracing algorithm for extracting ridges that also accurately captures ridge behavior at umbilics and ridge turning points. The algorithm traverses ridge segments by detecting ridge points while advancing and sliding in principal directions on a surface in a novel manner, thereby computing connected curves of ridge points. The output of the algorithm is a set of curve segments, some or all of which may be selected for other applications such as those mentioned above. The results of our technique are validated by comparison with results from previous research and with a brute-force domain sampling technique.

1. Introduction

Ridge curves mark important intrinsic features of the shape of a surface. The formal mathematical study of the role of ridges in geometry began with the research of Porteous [1] and was first emphasized for shape analysis by Koenderink [2]. Since then, ridges have proven valuable in a variety of applications spanning diverse domains. They are view-independent curves and more stable against surface deformation compared to other feature curves such as curvature lines, which makes them very useful for shape matching [3–6]. They are useful in visualization applications since they capture perceptually salient features of an object [7–9]. Other applications include freeform surface quality control [10] and geophysical analysis [11,12]. Fig. 1 shows ridges and special types of ridges called crests of a *B*-spline surface model of the upper part of a femur bone that have been computed using the technique presented in this paper.

A variety of approaches have been previously presented for computing ridges of discrete surface representations including polygonal meshes and images and implicit surface representations that are typically used to approximate discrete data. Estimating principal curvatures and their derivatives are major challenges for discrete surface representations. Tracing is relatively simpler and is typically performed by detecting crossings of ridges on the boundaries of mesh polygons or image voxels.

In contrast, principal curvatures and their derivatives can be computed exactly at specific points given a parametric surface representation with sufficient smoothness, but tracing ridges is more difficult. In addition, ridges exhibit complex behavior around umbilics where multiple segments may coalesce, or form loops. Umbilics represent important features on surfaces and have been used in shape matching [13]. Therefore, it is essential to compute ridges around umbilics accurately. There are relatively few methods that address ridge computation at umbilics on parametric surfaces.

The main contribution of this paper is a new algorithm for computing ridges on tensor product *B*-spline surfaces that also accurately captures ridge behavior at umbilics and other special points such as turning points. The algorithm traces ridges using localized curvilinear coordinate systems formed by principal curvature lines. Key results on computing principal curvature lines at umbilics, presented in [14], are utilized in the approach presented in this paper. The output of the algorithm is a set of ridge curve segments that are

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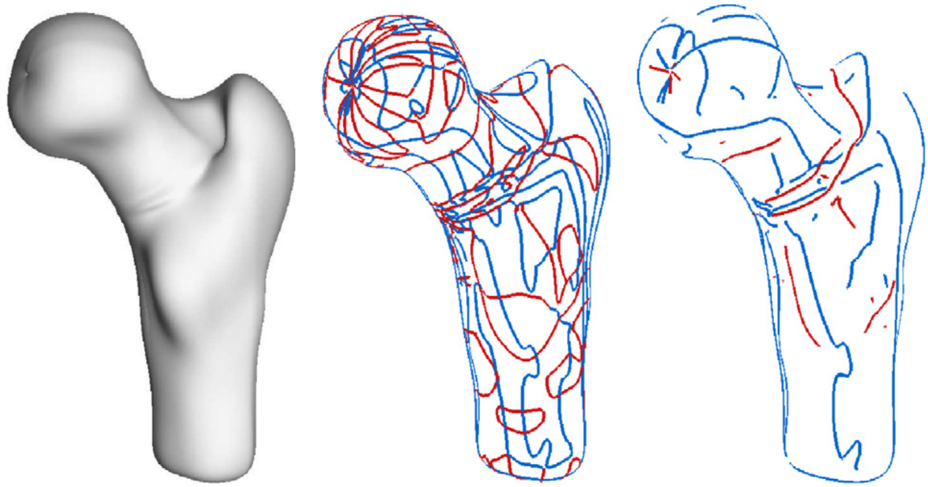

Fig. 1. Ridges and crests of a femur *B*-spline surface model.

available for use in other applications such as those mentioned previously. While the technique presented in this paper has been designed for tensor product piecewise rational parametric surfaces, it is quite general and extends to surface representations with multiple or trimmed patches with sufficient smoothness.

This paper also introduces a new type of geometrically salient region corresponding to major ridges. These salient regions also provide a means of measuring the importance of a ridge and its surrounding region in terms of higher order shape features. Earlier methods for quantifying the importance of ridges account for geometric properties only at ridge points [15–17]. The method introduced in this paper also considers the salient neighborhood of ridges. Salient regions provide additional information for studying geometric variation of similar shapes and are especially useful when the ridges themselves do not provide sufficient information. For example, the location of the ridges may be very similar across a set of similar surfaces but the geometry of regions surrounding the ridges may vary. An example is presented in Section 8 to illustrate this situation.

1.1. Definition and classification of ridges

There are a number of different definitions of ridge curves on surfaces with different meanings and intent. We follow the notation given in [18,19] and present that definition here. Consider a tensor product parametric surface $S(u, v) \in R^3$. Every point on $S(u, v)$, excluding umbilics, has two different principal curvatures ($\kappa_1 > \kappa_2$) and two corresponding principal directions (t_1, t_2).[1] Ridges are loci of points on a surface where one of the principal curvatures attains a critical value (i.e., local maximum, minimum or inflection) in its respective principal direction. This turns out to be equivalent to $\phi_i(u, v) = \langle \nabla\kappa_i, t_i\rangle = 0, i = 1$ or 2 (see [18, 20]). In this paper, we will henceforth refer to $\phi_i(u, v)$ as the *ridge condition* for the corresponding principal curvature.

Table 1 presents a classification of the various types of ridges. A ridge is called *elliptic* if κ_1 (κ_2) at a ridge point attains a local maximum (minimum) in the t_1 (t_2) direction, and termed *hyperbolic* otherwise. A *crest* is an elliptic ridge of the principal curvature with larger magnitude (see Table 1). The crest curve corresponding to the minimum principal curvature is typically called a *valley* or a *ravine*. It should be noted that some authors prefer to define ridges as the crest corresponding to the maximum principal curvature, while others refer to crests as κ_1-ridges, where $|\kappa_1| \geq |\kappa_2|$. In this paper, the term 'ridges' encompasses crests, elliptic, and hyperbolic ridges.

Table 1
Classification of ridges.

Ridge type	Definition
κ_1-ridge	$\nabla_{t_1}\kappa_1 \stackrel{def}{=} \langle\nabla\kappa_1, t_1\rangle = 0$
κ_2-ridge	$\nabla_{t_2}\kappa_2 \stackrel{def}{=} \langle\nabla\kappa_2, t_2\rangle = 0$
Elliptic ridge	$\nabla_{t_1}\kappa_1 = 0,\ t_1^T H_{\kappa_1} t_1 < 0$
	$\nabla_{t_2}\kappa_2 = 0,\ t_2^T H_{\kappa_2} t_2 > 0$
	$H_{\kappa_i} = \begin{bmatrix}\kappa_{iuu} & \kappa_{iuv} \\ \kappa_{iuv} & \kappa_{ivv}\end{bmatrix}, i = 1, 2$
Crest	$\nabla_{t_1}\kappa_1 = 0,\ t_1^T H_{\kappa_1} t_1 < 0,\ \lvert\kappa_1\rvert > \lvert\kappa_2\rvert$
(κ_2-crest → ravine or valley)	$\nabla_{t_2}\kappa_2 = 0,\ t_2^T H_{\kappa_2} t_2 > 0,\ \lvert\kappa_1\rvert < \lvert\kappa_2\rvert$

1.2. Generic properties of ridges

Various aspects of the behavior of ridges on surfaces are summarized in this section (see [2,18,19,21,22] for discussions and proofs). In this paper, only the generic case is considered.

1. Two different ridges of the same principal curvature do not cross each other, except at umbilics. This property reduces the complexity of ridge tracing significantly.
2. κ_1-ridges may cross κ_2-ridges at so-called *purple* points.
3. Ridges of a particular principal curvature do not have start or end points within a model (excluding the boundary of an open surface), except at umbilics.
4. Although principal directions are not defined at umbilics, ridges do occur at umbilics and exhibit complex behavior around umbilics.
5. Elliptic ridges, and therefore crests, do not occur at umbilics.
6. An umbilic may be classified as either a 1-ridge umbilic or a 3-ridge umbilic depending on the number of ridges arriving at the umbilic.
7. Ridges of a principal curvature intersect its corresponding principal direction transversally on the surface (R^3) except at a few isolated locations. This property enables tracing ridges on local coordinate systems formed by principal directions on a surface.

[1] t_1 and t_2 are 2D vectors denoting elements of the tangent plane at $S(u, v)$.