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Configuration products and quotients in geometric modeling

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ABSTRACT

The six-dimensional space $\mathbb{SE}(3)$ is traditionally associated with the space of configurations of a rigid solid (a subset of Euclidean three-dimensional space $\mathbb{R}^3$). But a solid itself can be also considered to be a set of configurations, and therefore a subset of $\mathbb{SE}(3)$. This observation removes the artificial distinction between shapes and their configurations, and allows formulation and solution of a large class of problems in mechanical design and manufacturing. In particular, the *configuration product* of two subsets of configuration space is the set of all configurations obtained when one of the sets is transformed by all configurations of the other. The usual definitions of various sweeps, Minkowski sum, and other motion related operations are then realized as projections of the configuration product into $\mathbb{R}^3$. Similarly, the dual operation of *configuration quotient* subsumes the more common operations of unsweep and Minkowski difference. We identify the formal properties of these operations that are instrumental in formulating and solving both direct and inverse problems in computer aided design and manufacturing. Finally, we show that all required computations may be approximated using a fast parallel sampling method on a GPU and provide error estimates for the approximation.

1. Introduction

1.1. Shape in configuration space

Swept sets (or sweeps) are one of the fundamental representation schemes in geometric and solid modeling [1]. Generally a swept solid S may be represented by a pair (A, B) of sets and a mapping $g : A \times B \to \mathbb{R}^3$, such that $S = g(A, B)$. Typically A is a one parameter family of rigid transformations, B is a point set in $\mathbb{R}^3$, and

$$g(A, B) = \mathtt{sweep}(B, A) = \bigcup_{a \in A} B_a$$

where B_a denotes set B transformed by a [2]. Most commercial CAD systems provide (limited) functionality for constructing swept solids, for example, in a form of a two-dimensional cross section moving on a space trajectory that is transversal to the plane of the cross section. The problems of constructing, approximating, and representing these and other types of sweeps have been studied extensively, e.g. see a survey in [3]. Most known methods assume a particular representation of the point set B and/or of the trajectory A, and tend not to be broadly applicable.

The Minkowski sum [4,5] of two subsets A, B of $\mathbb{R}^3$ is defined as the direct sum $A \oplus B$, with A and B being treated as a collection of (vector) positions. See Fig. 1 for an example. In this sense, the Minkowski sum can also be considered a sweep $g(A, B) = A \oplus B$, which is both generalized because A is no longer limited to a one-parameter family of transformations, and restricted because A contains only translations but does not allow any rotations. Notice also that the symmetry of Minkowski sum with respect to sets A and B is no longer obvious when it is viewed as a swept solid (but of course, it is still true). Despite being restricted to translational motions/configurations, Minkowski operations are frequently applied in motion planning [6,7], containment and packaging [8] layout [9], image processing [4] and many other graphics and shape modeling applications [5,10]. Their popularity is largely due to the rich algebraic structure that forms the foundation of mathematical morphology [4]. The same algebraic structure has been shown to exist for the traditional sweeps [11], reinforcing the close relationship between sweeps and Minkowski operations.

In order to unify various sweeps and Minkowski operations within a single, more general, and hence more powerful computational framework, we will consider a solid in terms of the positions *and* orientations associated with its points. This view is equivalent to specifying a set of coordinate frames at each point in the solid, thus implying the solid can be treated as a set of rigid transformations relative to an absolute coordinate system, and therefore as a subset of the six dimensional configuration space $\mathbb{R}^3 \times \mathbb{SO}(3)$ [7]. The extension removes the artificial distinction between shapes and their transformations because both are now subsets of the configuration space. Furthermore, this view leads to the generalization of a swept set.

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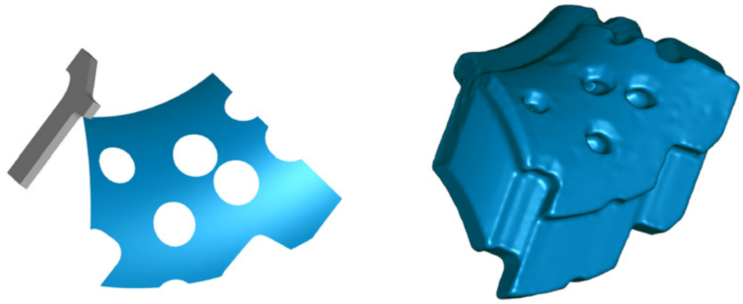

Fig. 1. Minkowski sum (shown on the right) of a solid and a surface (shown on the left) computed as a projection of the configuration product. Features from both input shapes can be seen in the Minkowski sum.

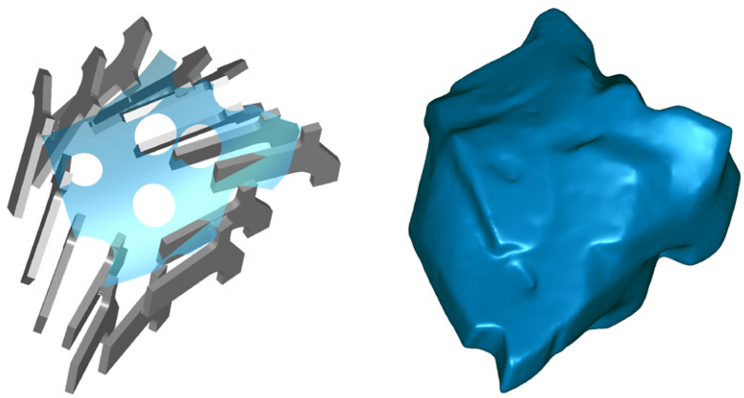

Fig. 2. Sweeping a solid over a surface computed as the projection of the configuration product of the two shapes. Left: the projection of the configuration product shown at discrete points on the surface boundary. Right: the projection of the configuration product corresponds to the sweep of the solid as it moves according to transformations defined by the surface.

Adopting the representation of configuration space as the Special Euclidean Group $\mathbb{SE}(3)$ [12] whose elements are 4×4 homogeneous transformation matrices, given a pair (A, B) of subsets of $\mathbb{SE}(3)$ the *configuration product* is a mapping $f : A \times B \rightarrow \mathbb{SE}(3)$ defined by

$$f(A, B) = A \otimes B = \bigcup_{a \in A,\ b \in B} a \cdot b$$

where $\cdot$ represents the group operation of matrix multiplication. Swept sets and Minkowski sums are then both sets of configurations $b \in B$ transformed by rigid transformations $a \in A$, and projected as point sets into the Euclidean space $\mathbb{R}^3$.

This paper argues that configuration product is a key geometric modeling operation that allows the formulation and solution of many problems in spatial design/planning involving relative configuration and/or motion constraints. Broadly, all such problems can be classified as either direct or inverse.

Direct problems usually require computing the 6D space of configurations occupied by an object B as it is transformed according to the set A. Such problems reduce to a direct evaluation and representation of the configuration product $A \otimes B$, and subsume the classical problems of computing various instances of Minkowski sums and sweeps. For example, in manufacturing applications, it is often desirable to compute the sweep of a solid (tool) as it moves over a curve or surface while maintaining a particular orientation with respect to the curve/surface normals/tangents. An example is shown in Fig. 2. Another example of a direct problem is the determination of a mechanism's (e.g. robot's) workspace, where it is required to explicitly compute all the positions and orientations achievable by a mechanism. It is common to distinguish between reachable (position) and dextrous (orientation) workspaces [13], but both are special cases of the configuration product.

Inverse problems typically impose the constraint $A \otimes B \subseteq C$, where C is a given subset of $\mathbb{SE}(3)$, and require computing the largest possible set of configurations A or the largest shape B that satisfies the constraint. In this sense, the inverse problems define A or B implicitly, and include common problems of packaging, where the set B must fit inside C under a set of transformations A, and motion planning where C plays the role of free (configuration) space. Formally, the inverse problems can be solved using operations dual to configuration product called *configuration quotients*, which are proper generalizations of the Minkowski difference and unsweep operations [11]. They are defined and studied in Section 2 of the paper.

Several of the direct and inverse problems described above, such as sweeps over manifolds (curves and surfaces), design of maximal shapes under arbitrary motion constraints, and determination of maximal transformations of shapes to satisfy containment constraints, are difficult to formulate and solve except in special cases. The main contribution of this paper is to show that all these problems and many others involving general motions and relative configurations of solids may be effectively formulated using configuration products and quotients. We show that products and quotients may be rapidly approximated by sampling subsets of configuration space and computing all pairwise multiplications between the sampled sets. The inherently parallel computational procedure is mapped onto the GPU architecture where the configuration products can be computed at a fraction of the computational cost associated with a similar sampling algorithm on the CPU. We also derive sampling error estimates that can be used to develop effective sampling strategies and to measure deviations from exact computations.

1.2. Paper outline

Basic properties of configuration products and quotients are summarized in Section 2. The duality between configuration products and quotients subsumes the well known duality relationship