Computer Aided Geometric Design 29 (2012) 363–378

Contents lists available at SciVerse ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Matching admissible G^2 Hermite data by a biarc-based subdivision scheme ☆

Chongyang Deng [a,b], Weiyin Ma [b,*]

[a] Institute of Applied Mathematics and Engineering Computations, Hangzhou Dianzi University, Hangzhou 310018, China
[b] Department of Mechanical and Biomedical Engineering, City University of Hong Kong, Kowloon, Hong Kong, China

ARTICLE INFO

Article history:
Received 15 October 2010
Received in revised form 18 March 2012
Accepted 20 March 2012
Available online 29 March 2012

Keywords:
Geometry driven subdivision
Nonlinear subdivision scheme
Admissible G^2 Hermite interpolation
Spiral
Monotone curvature
Shape preserving

ABSTRACT

Spirals are curves with single-signed, monotone increasing or decreasing curvature. A spiral can only interpolate certain G^2 Hermite data that is referred to as admissible G^2 Hermite data. In this paper we propose a biarc-based subdivision scheme that can generate a planar spiral matching an arbitrary set of given admissible G^2 Hermite data, including the case that the curvature at one end is zero. An attractive property of the proposed scheme is that the resulting subdivision spirals are also offset curves if the given input data are offsets of admissible G^2 Hermite data. A detailed proof of the convergence and smoothness analysis of the scheme is also provided. Several examples are given to demonstrate some excellent properties and practical applications of the proposed scheme.

1. Introduction

Curves with monotone curvature are called spirals. It is widely accepted that a planar curve has a pleasing look and is a fair curve if it has a curvature plot with relatively few regions of monotonically varying curvature (Farin, 1997), i.e., if it has a small number of spiral pieces. It is natural to consider the problem of interpolating two-point G^2 Hermite data by a spiral due to the requirement of achieving G^2 continuity at the joins of neighboring spirals (Goodman et al., 2009). The results can be used for certain applications, such as in highway designs, railway routes, automotive and naval designs, and aesthetic applications (Burchard et al., 1994; Gibreel et al., 1999; Habib and Sakai, 2007a, and references therein).

A spiral can only interpolate certain Hermite data (Guggenheimer, 1963). Such data is referred to as *admissible G^2 Hermite data*. Matching admissible G^2 Hermite data by a spiral is extensively studied in the literature. Meek and Walton form a spiral from a clothoid and additional circular arcs that can match any admissible G^2 Hermite data (Meek and Walton, 1992) from a segment of a given spiral together with circular arcs (Meek and Walton, 1998). Ait Haddou and Biard (1995) use spirals formed from pairs of involutes of a Pythagorean hodograph curve (PH curve) to interpolate any admissible G^2 Hermite data. Kuroda and Mukai (2000) use spirals formed from the involutes of circular arcs to construct G^2 Hermite interpolates. Recently, much research focuses on designing spirals using (rational) Bézier curves (Mineur et al., 1998; Frey and Field, 2000; Dietz and Piper, 2004; Dietz et al., 2007) and Pythagorean hodographs (Farouki, 1997; Walton and Meek, 1996, 2004) that are used to design transition curves (Habib and Sakai, 2005, 2007a, 2007b; Walton and Meek, 2002, 2007). Different from the above-mentioned methods, in this paper, we propose a subdivision scheme that generates a spiral matching any admissible G^2 Hermite data.

☆ This paper has been recommended for acceptance by M.A. Sabin.

* Corresponding author.
E-mail addresses: dcy@hdu.edu.cn (C. Deng), mewma@cityu.edu.hk (W. Ma).

0167-8396/$ – see front matter
http://dx.doi.org/10.1016/j.cagd.2012.03.010

Generating smooth curves by repeated subdivision is an important modeling method in computer aided geometric design. Compared with other modeling methods, subdivision schemes are simpler and more efficient, but how to control the shape of the limit curve is a very difficult problem. For example, for linear subdivision schemes such as the four point subdivision scheme (Dyn et al., 1987) and its extensions (Hassan et al., 2002), there are often artifacts and undesired inflexions on the limit curves (Marinov et al., 2005). The limit curves of nonlinear (Aspert et al., 2003) and geometric driven schemes (Dyn et al., 1992; Marinov et al., 2005; Yang, 2006) are usually only G^1 continuous. Recently, Deng and Wang (2010) propose an incenter subdivision scheme for curve interpolation. By controlling the discrete tangent and discrete curvature accurately, the limit curves possess good properties such as shape preserving, G^2 continuity and fair conditions. The incenter scheme can generate a spiral from two-point G^1 Hermite data. However, the spiral generated by the incenter scheme cannot match admissible G^2 Hermite data to be addressed in this paper.

For a planar spiral, following Kneser's Theorem (Guggenheimer, 1963), we know that the osculating circle (circle of curvature) of the point with large curvature is inside the osculating circle of the point with small curvature. Based on the above fact, for each edge, we select a new point and its tangent vector and curvature as well such that its osculating circle is inside the osculating circle of one end point and outside that of the other. In theory, there are many points as well as their tangents and curvatures which satisfy this condition. The crux lies in how to select new points with their tangents and curvatures by a comparatively simple way and produce a pleasing G^2 continuous limit curve. In this paper, we propose a biarc-based subdivision scheme that produces a G^2 spiral spline interpolating a given set of admissible G^2 Hermite data.

Using a G^1 biarc curve to interpolate or approximate points, lines or arbitrary curves, is extensively studied in the literature because of the simplicity of biarc splines and the capability of milling machines to move along straight lines and circular paths. With an appropriate method to choose the free parameters of biarcs, a planar spiral can be approximated by a spiral arc spline (Meek and Walton, 1999). This is the main reason for us to use a biarc-based scheme for inserting new points, which results in simple subdivision rules while meeting the required interpolation conditions.

In highway designs, railway routes and related applications, it is usually required to produce an offset of a spiral. Similar to the definition of offset curve, we define the offset of G^2 Hermite data, called offset G^2 Hermite data, and prove that all the offset G^2 Hermite data of admissible G^2 Hermite data are also admissible G^2 Hermite data. We also define degenerate admissible G^2 Hermite data for admissible G^2 Hermite data and show that an admissible G^2 Hermite data and all of its offset G^2 Hermite data correspond to the same degenerate admissible G^2 Hermite data. In the refinement step of the biarc-based subdivision scheme proposed in this paper, we first map the admissible G^2 Hermite data to its degenerate admissible G^2 Hermite data. We then calculate the provisional point, tangent and curvature and map it back to derive the newly inserted point, tangent and curvature. As to be proved in this paper, it ensures that the resulting spiral generated from the offset G^2 Hermite data is an exact offset of the spiral generated from the given admissible G^2 Hermite data. This is an excellent property of the proposed subdivision scheme for both the above mentioned and many other applications.

The rest of this paper is organized as follows. In Section 2 we will introduce the background of the admissible G^2 Hermite data and biarc. The subdivision scheme is presented in Section 3. The smoothness analysis of the scheme is presented in Section 4. In Section 5 we provide some experimental examples to demonstrate the excellent properties and practical applications of the proposed scheme. Section 6 is devoted to the conclusions and future work of the paper.

2. Notations and background

2.1. Notations

In this paper, we define an angle that is positive if it is counter-clockwise and negative otherwise. Because spirals are curves of single-signed curvature, in this paper, we assume that all the angles are positive. For a vector $V = (V_x, V_y)^T$, the rotated vector with angle α from V can be obtained as $R(\alpha)V$, where $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is the rotation matrix. The Euclidean distance $\|p_i^k p_{i+1}^k\|$ between p_i^k and p_{i+1}^k is $\|p_{i+1}^k - p_i^k\|$.

Two-point G^2 Hermite data is represented by its end points, tangents and curvatures as $\{A, T_A, \kappa_A; B, T_B, \kappa_B\}$ or by its end points, tangents and centers of osculating circles as $\{A, T_A, O_A; B, T_B, O_B\}$. The radii of the osculating circles at A and B are defined as $r_A = 1/\kappa_A$ and $r_B = 1/\kappa_B$, respectively. For G^1 Hermite data, the angles from T_A to AB, from AB to T_B are denoted as α and β, respectively (see Fig. 1).

2.2. Admissible G^2 Hermite data

For admissible G^2 Hermite data, a detailed definition can be found in Goodman et al. (2009). Combining Kneser's Theorem and Vogt's Theorem (Guggenheimer, 1963), we give a more intuitive description of the admissible G^2 Hermite data:

Proposition 2.1. *For G^2 Hermite data $\{A, T_A, O_A; B, T_B, O_B\}$ with $\alpha > \beta$, there exists a matching spiral if and only if $\|O_A - O_B\| < r_B - r_A$ (see Fig. 1).*

Remark 2.2. A line can be considered as a special case of circle with infinite radius. If the curvature of B is 0, the osculating circle of B becomes the tangent line BT_B and the condition $\|O_A - O_B\| < r_B - r_A$ is replaced by an alternative condition