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The Bernstein polynomial basis: A centennial retrospective ☆

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A R T I C L E   I N F O

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A B S T R A C T

One hundred years after the introduction of the Bernstein polynomial basis, we survey the historical development and current state of theory, algorithms, and applications associated with this remarkable method of representing polynomials over finite domains. Originally introduced by Sergei Natanovich Bernstein to facilitate a constructive proof of the Weierstrass approximation theorem, the leisurely convergence rate of Bernstein polynomial approximations to continuous functions caused them to languish in obscurity, pending the advent of digital computers. With the desire to exploit the power of computers for geometric design applications, however, the Bernstein form began to enjoy widespread use as a versatile means of intuitively constructing and manipulating geometric shapes, spurring further development of basic theory, simple and efficient recursive algorithms, recognition of its excellent numerical stability properties, and an increasing diversification of its repertoire of applications. This survey provides a brief historical perspective on the evolution of the Bernstein polynomial basis, and a synopsis of the current state of associated algorithms and applications.

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1. Introduction

At their inception, it is extremely difficult to predict the subsequent evolution and ultimate significance of novel mathematical ideas. Concepts that at first seem destined to revolutionize the scientific landscape — such as Hamilton's *quaternions* — may gradually lapse into relative obscurity (Crowe, 1967). On the other hand, methods introduced to facilitate theoretical proofs of seemingly limited scope and practical interest may eventually flourish into useful tools that gain widespread acceptance in diverse practical computations.

The latter category undoubtedly includes the *Bernstein polynomial basis*, introduced 100 years ago (Bernstein, 1912) as a means to constructively prove the ability of polynomials to approximate any continuous function, to any desired accuracy, over a prescribed interval. Their slow convergence rate, and the lack of digital computers to efficiently construct them, caused the Bernstein polynomials to lie dormant in the *theory* rather than *practice* of approximation for the better part of a century.[1] Ultimately, the Bernstein basis found its true vocation not in approximation of functions by polynomials, but in exploiting computers to interactively design (vector-valued) polynomial functions — i.e., *parametric curves and surfaces*. In this context, it became apparent that the Bernstein coefficients of a polynomial provide valuable insight into its behavior over a given finite interval, yielding many useful properties and elegant algorithms that are now being increasingly adopted in other application domains.

The centennial anniversary of the introduction of the Bernstein basis is an opportune juncture at which to survey and assess the attractive features and algorithms associated with this remarkable representation of polynomials over finite domains, and its diverse practical applications. It seems probable that Bernstein would be amazed to witness the widespread interest — albeit in rather different contexts — that a simple but powerful idea in the paper (Bernstein, 1912) has elicited, one hundred years after its first appearance.

The intent of this paper is: (1) to provide a historical retrospective on the introduction and evolution of the Bernstein basis as a practical computational tool; (2) to succinctly survey, as a guide to the uninitiated, the key properties and algorithms associated with it; and (3) to briefly enumerate the current variety of applications in which it has found use. The plan for the remainder of the paper is as follows. The academic career of Sergei Natanovich Bernstein is briefly summarized in Section 2, followed by a discussion of his constructive proof of the Weierstrass approximation theorem in Section 3. Section 4 then describes the contributions of the French engineers Pierre Bézier and Paul de Faget de Casteljau, in promoting the use of the Bernstein basis in the context of computer-aided design for the automotive industry during the 1960s and 1970s. Some characteristic features of the Bernstein basis, upon which many useful properties and elegant algorithms are based, are identified in Section 5, while Section 6 describes a fundamental feature of the Bernstein form, that was not fully appreciated until the 1980s: its *numerical stability* with respect to coefficient perturbations or floating-point round-off errors. A synopsis of alternative approaches and interpretations is presented in Section 7 — the *shift operator*; the theory of "blossoming" or *polar forms*; connections with probability theory; and methods based on generating functions and discrete convolutions. Section 8 addresses the central role of the Bernstein form as a cornerstone of *computer-aided geometric design*, while Section 9 summarizes its applications as a basic computational tool in other scientific/engineering fields. Finally, Section 10 assesses the current status and future prospects of the Bernstein representation of polynomials over finite domains.

[1] "In theory, there is no difference between theory and practice. In practice, there is." Attributed to Yogi Berra.