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Simple determination via complex arithmetic of geometric characteristics of Bézier conics ☆

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ABSTRACT

We show how to compute in a straightforward manner the geometric characteristics of a conic segment in rational Bézier form, by employing complex arithmetic. For a central conic, a simple quadratic equation defines the foci location, and its solution furnishes not only an explicit formula for the foci, but also for the center, axis direction and linear eccentricity.

1. Introduction

A key reason for adopting the rational model in CAGD is its ability to represent exactly conic sections, or conics for short. Due to their remarkable reflective properties, these curves find widespread use in optical and telecommunication instruments (Downs, 1993). The quadratic Bézier representation of conics is thus found in most textbooks on CAGD (Farin, 2001; Farin and Hansford, 2000; Hoschek and Lasser, 1993), and all NURBS monographs (Farin, 1999; Piegl and Tiller, 1997; Rogers, 2001). Nevertheless, only Piegl and Tiller (1997) include formulae, due to Lee (1987), for obtaining the geometric characteristics (center, foci, axes, ...) of an already constructed rational quadratic Bézier segment, in terms of its weights and control points. Recently, Xu et al. (2010) have derived explicit formulae, based on Lee's results, for computing the eccentricity of a Bézier conic.

Though not difficult, Lee's computations are involved, requiring for instance the use of Lagrange multipliers to derive the axis length. Furthermore, no simple expressions are given for the foci, the relevant points regarding reflective properties. This shortcoming was tackled by Albrecht (2001), who emphasizes determining the foci of a given conic in Bézier form. However, her derivation involves computing the singular points of a certain algebraic curve of degree four. Explicit formulae for all the geometric characteristic are found in the article by Goldman and Wang (2004), although they do not employ the customary Bézier representation. They derive their results from the invariants of rational quadratic parameterizations under rational linear reparameterizations. Finally, Cantón et al. (2011) show how to compute the geometric characteristics of a conic in Bézier form, by writing its implicit equation in coordinate-free fashion.

We present here an alternative and more geometric approach, based on representing conics in the complex plane $\mathbb{C}$. The space $\mathbb{C}$ enjoys the algebraic structure of *field*, where not only can points be added, but also multiplied and divided, and square roots are meaningful. Complex arithmetic drastically simplifies the expressions for the foci, center, and linear eccentricity of a Bézier conic. In Section 2, we first characterize the focus in a trivial form with complex notation. This basic

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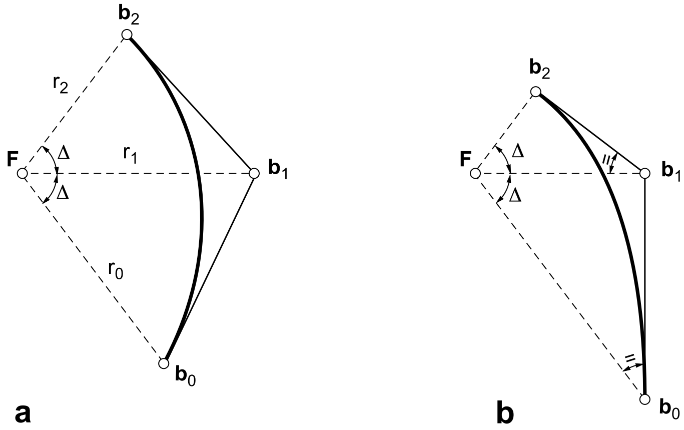

Fig. 1. Bézier points $\mathbf{b}_k$ of a conic with focus at $\mathbf{F}$: a) General conic. b) Parabola.

result allows us to obtain the foci in Section 3, as the solutions of a (complex) quadratic equation. The center and linear eccentricity come as by-products. Finally, conclusions are drawn in Section 4.

2. Characterizing the foci with complex products

Before trying to obtain the foci of a given Bézier conic, we recall the inverse problem, that is, how to construct arbitrary Bézier conics of given focus $\mathbf{F}$, say in standard form (with unit weights for the endpoints $\mathbf{b}_0$, $\mathbf{b}_2$). Sánchez-Reyes (2004) shows that, whereas we can choose arbitrarily $\mathbf{b}_0$, $\mathbf{b}_2$, the inner point $\mathbf{b}_1$ and weight $w_1 = w$ are constrained:

(1) The point $\mathbf{b}_1$ lies on the bisector of the lines $\mathbf{Fb}_0$, $\mathbf{Fb}_2$, i.e., so that the segments $\mathbf{b}_0\mathbf{b}_1$ and $\mathbf{b}_1\mathbf{b}_2$ see $\mathbf{F}$ with the same angle Δ (Fig. 1a).
(2) The inner weight w takes a specific value, determined by the radial distances r_k:

$$w^2 = \frac{r_0 r_2}{r_1^2}, \quad r_k = |\mathbf{r}_k|, \quad \mathbf{r}_k = \mathbf{b}_k - \mathbf{F}. \tag{1}$$

Condition (1) simply rewrites in Bézier representation a classical result (Salmon, 1960), which states that the intersection of any two tangents to a conic and both points of contact are seen from $\mathbf{F}$ within equal angles Δ.

The radial values $\mathbf{r}_k$ can hence be written in polar form as complex exponentials $\mathbf{r}_k = r_k e^{i\theta_k}$ (Needham, 1997) of moduli r_k, and arguments θ_k equally spaced by an angle Δ. By introducing these complex exponentials and the value w (1), we obtain a startling simple characterization of the focus $\mathbf{F}$, in terms of complex products:

$$(w\mathbf{r}_1)^2 = \mathbf{r}_0\mathbf{r}_2, \quad \mathbf{r}_k = \mathbf{b}_k - \mathbf{F}. \tag{2}$$

For the case of a parabola ($w = 1$), this relationship indicates that the values $\mathbf{r}_k$ form a geometric progression, which admits an intuitive interpretation: the adjacent triangles $\mathbf{Fb}_0\mathbf{b}_1$ and $\mathbf{Fb}_1\mathbf{b}_2$ are similar (Fig. 1b). This geometric property was already noted by Sánchez-Reyes (1990), and also derives from the pedal-point construction of a parabola (Ueda, 1997).

3. Computing the foci, center and linear eccentricity

Suppose that we are given a Bézier conic in standard form, with control points $\mathbf{b}_k$ and inner weight $w_1 = w$. To find the focus $\mathbf{F}$, simply interpret equality (2) as an equation in the unknown $\mathbf{F}$, and solve it. As shown in this section, simple algebra yields the roots, according to the well-know case distinction that determines the conic type: ellipse or hyperbola ($w \neq 1$), and parabola ($w = 1$). To fix our ideas, we assume the customary condition $w > 0$. However, the sign of w plays no role, as reflected in the characterization (2), where w is squared. By reversing its sign, we just obtain the complementary segment of the same conic (Farin, 2001).

3.1. Ellipse or hyperbola ($w \neq 1$)

The case $w \neq 1$ yields a *central conic*, i.e., an ellipse ($w < 1$) or hyperbola ($w > 1$). Eq. (2) is quadratic in $\mathbf{F}$ and, after straightforward manipulation, can be written in monic form:

$$\mathbf{F}^2 - 2\mathbf{CF} + \mathbf{d} = 0, \tag{3}$$

with coefficients $\mathbf{C}$, $\mathbf{d}$ expressible as barycentric combinations: