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Computing lines of curvature for implicit surfaces

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ABSTRACT

Lines of curvature are important intrinsic characteristics of a curved surface used in a wide variety of geometric analysis and processing. Although their differential attributes have been examined in detail, their global geometric distribution and topological pattern are very difficult to compute over the whole surface because of umbilical points and unstable numerical computation. No studies have yet been carried out on this problem, especially for an implicit surface. In this paper, we present a scheme for computing and visualizing the lines of curvature defined on an implicit surface. A key structure is introduced, conveying significant structure information about lines of curvature to facilitate their investigation, rather than computing their whole net. Our current framework is confined to a collection of manageable structures, consisting of algorithms to locate some seed umbilical points, to compute the lines of curvature through them, and finally to assemble this structure. The numerical implementations are provided in detail and a novel evaluation function measuring the violation of umbilical points in an implicit surface, i.e. indicating how much a point is to be umbilical, is also presented. This paper is the continuation of [Che, W.J., Paul, J.-C., Zhang, X.P., 2007. Lines of curvature and umbilical points for implicit surfaces. Computer Aided Geometric Design 24 (7), 395–409].

1. Introduction

A line of curvature is one of the most important geometric attributes for a curved surface, indicating a direction field for the extreme curvature across the surface. Together with the principal curvatures, lines of curvature (LOC) provide a complete description of variations in curvature of the surface. They are useful tools in surface analysis for exhibiting variations of the principal directions. LOC can guide the analysis of surfaces, widely used in geometric design, shape recognition, polygonization of surfaces, surface rendering etc. (Alliez et al., 2003; Beck et al., 1986; Hallinan et al., 1999; Maekawa et al., 1996; Patrikalakis and Maekawa, 2002).

1.1. Background

Given a point **p** on a smooth manifold surface **S** in 3D Euclidean space, the normal curvature in a tangent direction at **p** is the curvature of the intersection curve of **S** with the plane generated by the tangent vector and the normal to **S** at **p**. The curvature depends on the tangent vector. The principal curvatures are the extreme values of normal curvature –

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maximum or minimum, and their corresponding directions are principal directions. If both principal curvatures are equal, **p** is an umbilical point and every tangent vector can be thought of as a principal direction.

The integral curve of maximal (resp. minimal) principal direction field is called a line of maximal (resp. minimal) curvature. Two principal directions are uniquely determined and are orthogonal at **p** away from an umbilical point. Therefore, there are always two LOC through non-umbilical **p** and they cross at a right angle. On **S** without umbilical points, LOC constitute an orthogonal net, but the existence of umbilical points destroys its elegance – the net patterns are disturbed significantly and could become very complex, leading to a breakdown in the regularity of orthogonality. Therefore, an umbilical point is a singularity with anomalous behavior in the LOC net.

The set of umbilical points is denoted by $\mathcal{U}$; the family of maximal (resp. minimal) LOC by $\mathcal{F}_i$, $i = 1, 2$, and called the maximal (resp. minimal) principal foliation of **S**. The triple $\mathcal{P} = (\mathcal{U}, \mathcal{F}_1, \mathcal{F}_2)$ is the principal configuration of **S**. Historically, principal configuration has been studied from the angle of differential geometry. Its differential geometry is usually based on the parametric representation of surfaces. The study of principal configuration locally around an umbilical point has received considerable attention. See Gutiérrez and Sotomayor (1990), Porteous (2001), Gutiérrez et al. (2004), Sotomayor and Garcia (2007) and references therein. The global structure of principal configuration, however, has been known only for very rare classical examples (Gutiérrez and Sotomayor, 1990). It is difficult to calculate the LOC net to visualize the principal configuration correctly in contrast to principal curvatures and those work provide no methods to obtain it globally. Although there has been some related studies, they mainly concentrate on local computation and explain little about computing the global behavior of LOC over the surface, especially on an implicit surface. To the best of our knowledge, no algorithm is yet able to describe any global differential pattern of LOC with some guarantee. Even their computation with $\mathcal{U} = \emptyset$ is still impossible.

LOC are difficult to obtain because the reconstructed line, with simple numerical integration techniques, diverges from the true one, and the algorithm never ends at the expected point due to computational errors. Relaxation algorithms have been proposed for a closed line, but they can work only if there is some information on how to place an initial closed curve in the surface (Thirion, 1996). An alternative is to generate the field of principal directions via visualization technologies of vector field over its discretization by sampling the surface, but it usually gives us an ambiguous observation for surfaces with a high variation in curvature unless the sampling is dense enough. In order to represent the vector field by its streamlines, i.e. integral curves, we have to choose a set of starting points on the surface in such a way that the density of the points is proportional to the intensity of the field, and then trace each starting point in both positive and negative directions to generate the streamlines, going back to the way by using numerical integration methods.

Generally speaking, two aspects need to be taken into account to give an accurate description of $\mathcal{P}$. The first involves the geometric distribution of $\mathcal{F}_i$, $i = 1, 2$, and the second is the topological patterns among $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{U}$. Their uneven distribution throughout the surface results in the difficulties of robust computation, in particular near an umbilical point. Few studies have dealt with LOC defined on an implicit surface. The primary goal of our paper is to attempt to develop numerical computation techniques for the automatic generation of LOC on an implicit surface.

1.2. Previous research

Umbilical points are attributes closely related to LOC. The earliest classification of umbilical points and the generic features of LOC near an umbilical point are due to Darboux (1896) (Maekawa et al., 1996; Porteous, 2001; Sotomayor and Garcia, 2007). Since then, many authors have carried out much research on them, most of which was targeted at local differential properties (Gutiérrez and Sotomayor, 1990; Gutiérrez et al., 2004; Maekawa et al., 1996; Porteous, 2001). The study of the global features of $\mathcal{P}$ has also received considerable attention, and see Sotomayor and Garcia (2007) for references. No studies, however, have considered aspects of numerical computation, which are important to understand a given surface in practice since a priori computation is usually infeasible.

For a parametric surface, especially for a general free-form surface, some previous work does exist which has investigated the numerical computation of LOC. These methods usually consider two coupled, non-linear differential equations in terms of two parametric coordinates respectively, resulting in a numerical method for a B-spline surface (Beck et al., 1986; Farouki, 1998; Maekawa et al., 1996). Maekawa and Patrikalakis (1994) described a computational method to locate isolated umbilical points on a parametric polynomial surface. But how to derive the LOC net with umbilical points was not fully discussed in these studies.

Little work has been carried out on other surface representations. Alliez et al. (2003) reduced LOC calculation on a triangular mesh to a 2D parametric plane by flattening the surface via a discrete conformal parameterization. The umbilical points were found by going over each triangle and solving a 2×2 linear system in the tensor field. Their method is still based on the parametric surface form and as some LOC information could be lost, it fails to fit accurately to the computation. Che et al. (2007) studied the explicit representation of principal directions defined on an implicit surface, based upon which the differential equation of LOC and a new criterion for umbilical points were obtained, but their numerical implementation was not carried out. To our knowledge, how to locate umbilical points on an implicit surface has not yet been reported in literature.

It is impossible to cover the surprising richness of differential geometry on LOC and umbilical points. The interested readers can refer to related work and textbooks (Beck et al., 1986; Che et al., 2007; Maekawa et al., 1996; Porteous, 2001), and also references therein for basic facts. The study of LOC and their umbilical singularities is described in Gutiérrez and