Available online at www.sciencedirect.com
ScienceDirect
Computer Aided Geometric Design 24 (2007) 79–89
COMPUTER AIDED GEOMETRIC DESIGN
www.elsevier.com/locate/cagd

Modeling sphere-like manifolds with spherical Powell–Sabin B-splines

Jan Maes *, Adhemar Bultheel

Department of Computer Science, K.U. Leuven, Celestijnenlaan 200A, B-3001 Heverlee, Belgium

Received 26 April 2006; received in revised form 30 October 2006; accepted 2 November 2006

Available online 6 December 2006

Abstract

We present an algorithm for computing a B-spline representation for Powell–Sabin splines on the sphere. The B-splines form a partition of unity and we define control points that constitute control triangles that give us a good insight in the shape of the spline. We further consider a number of CAGD applications such as approximation and compression of a given sphere-like triangular mesh, and editing global shape and local detail of a spline using spline subdivision.

MSC: 65D07; 65D17; 68U05

Keywords: Spherical Powell–Sabin splines; B-splines; Control triangles; Spherical surfaces

1. Introduction

Modeling sphere-like manifold surfaces is a challenging task. In general they can be described as surfaces that are homeomorphic to the sphere, i.e., there exists a continuous invertible mapping from the sphere-like manifold onto the sphere. The traditional approach decomposes the geometric data into a group of charts and maps each chart into a planar parametric domain. Then a spline surface is fitted to each chart and the different patches are stitched together while maintaining some weak form of continuity such as tangent plane continuity, see, e.g., Eck and Hoppe (1996). The topology of each chart is restricted: it must be homeomorphic to a disk.

In this paper we describe a different approach. By using spherical Powell–Sabin (PS) B-splines on a spherical triangulation we can model spherical surfaces without decomposing them into different charts and mapping each chart to a planar parametric domain. The spherical PS B-splines have the same continuity properties as their planar counterparts and their construction is based on the concept of spherical Bernstein–Bézier polynomials that was introduced by Alfeld et al. (1996a). Furthermore the PS B-splines form a partition of unity and we find control points for the PS B-splines that can give us a good insight in the shape of the spline. We note that related work has been done by Pfeifle and Seidel (1995) and Neamtu (1996) who have extended the concept of simplex splines to the sphere.

* Corresponding author.
E-mail addresses: Jan.Maes@cs.kuleuven.be (J. Maes), Adhemar.Bultheel@cs.kuleuven.be (A. Bultheel).

0167-8396/$ – see front matter
doi:10.1016/j.cagd.2006.11.001

The paper is organized as follows. We begin by giving the definition of spherical Bernstein–Bézier polynomials and by reviewing how the spherical spline space of PS splines is constructed in Section 2. Next, in Section 3, we introduce B-splines for the space of spherical PS splines that form a partition of unity and we introduce control points and control triangles that give us insight in the shape of the spline. Section 4 is devoted to applications in CAGD and we show how these B-splines may be used in practice to approximate a given sphere-like triangular mesh. Lastly, we conclude with some final remarks.

2. Spherical Powell–Sabin splines

We begin by introducing homogeneous and spherical spline spaces following Alfred et al. (1996a, 1996b, 1996c). A function f defined on $\mathbb{R}^3$ is *homogeneous of degree d* provided that $f(\alpha v)=\alpha^d f(v)$ for all real α and all $v\in\mathbb{R}^3$. We are interested in the space $\mathbb{H}_d$ of *trivariate polynomials of degree d that are homogeneous of degree d*. The space $\mathbb{H}_d$ is a $\binom{d+2}{2}$ dimensional subspace of the space of trivariate polynomials of degree d. Let $\{v_1,v_2,v_3\}$ be a set of linearly independent unit vectors in $\mathbb{R}^3$. We call

$$\mathcal{T}:=\left\{v\in\mathbb{R}^3 \mid v=b_1(v)v_1+b_2(v)v_2+b_3(v)v_3 \text{ with } b_i(v)\geqslant 0\right\}$$

the *trihedron* generated by $\{v_1,v_2,v_3\}$. Each $v\in\mathbb{R}^3$ can be written in the form

$$v=b_1(v)v_1+b_2(v)v_2+b_3(v)v_3, \tag{1}$$

and we call $b_1(v), b_2(v), b_3(v)$ the *trihedral coordinates of* v *with respect to* $\mathcal{T}$. Given an integer $d\geqslant 0$, the *homogeneous Bernstein basis polynomials of degree d on* $\mathcal{T}$ are the polynomials

$$B^d_{ijk}(v):=\frac{d!}{i!j!k!}b_1(v)^i b_2(v)^j b_3(v)^k,\quad i+j+k=d,$$

and they form a basis for $\mathbb{H}_d$. We define a *spherical triangle* as the restriction of a trihedron $\mathcal{T}$ to the unit sphere S. The restrictions of the trihedral coordinates (1) to a spherical triangle with vertices v_1, v_2 and v_3 are called *spherical barycentric coordinates*. Any homogeneous polynomial p of degree d and its restriction to a spherical triangle τ has a Bernstein–Bézier representation with respect to τ

$$p(v):=\sum_{i+j+k=d} c_{ijk}B^d_{ijk}(v),$$

and the coefficients c_{ijk} are the Bézier ordinates.

We write $\mathbb{H}_d(\Omega)$ for the restriction of $\mathbb{H}_d$ to any subset Ω of the unit sphere S, and refer to $\mathbb{H}_d(\Omega)$ as the *space of spherical polynomials of degree d*. Similarly, we write $\mathbb{H}_d(H)$ for the restriction of $\mathbb{H}_d$ to any hyperplane H in $\mathbb{R}^3$. This is just the well-known space of bivariate polynomials. All these spaces have the same dimension $\binom{d+2}{2}$. Let Δ be a conforming spherical triangulation of $\Omega\subset S$. We recall that a triangulation is called conforming if two adjacent triangles share exactly one common vertex or one common edge. Then we define the *space of spherical splines of degree d and smoothness r associated with* Δ to be

$$S^r_d(\Delta):=\left\{s\in C^r(S)\colon\ s|_\tau\in\mathbb{H}_d(\tau),\ \tau\in\Delta\right\},$$

where $s|_\tau$ denotes the restriction of s to the spherical triangle τ. Keeping up continuity conditions between neighbouring spherical triangles results in nontrivial relations between their Bézier ordinates. Therefore we will focus on the *Powell–Sabin* 6*-split* of a triangulation to overcome this problem.

Suppose that the spherical triangulation Δ consists of triangles τ_j, $j=1,\dots,t$, with vertices v_i, $i=1,\dots,n$. For the remainder of the paper we will always assume that the diameter of the triangles in Δ is bounded,

$$\operatorname{diam}(\tau_j)\leqslant 1,\qquad \operatorname{diam}(\tau_j):=\sup\left\{\arccos(u\cdot v),\ u,v\in\tau_j\right\}.$$

This assumption guarantees that the projections that we use throughout are always well defined. The Powell–Sabin 6-split Δ^{PS} of Δ divides each triangle $\tau_j\in\Delta$ into six smaller triangles with a common vertex as follows:

(1) Define the interior point z_j for each triangle τ_j as the incenter of the triangle τ_j. If v_1, v_2, v_3 are the vertices of τ_j then we define its incenter as the point on S that is obtained by radially projecting the incenter of the planar