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On pseudo-harmonic barycentric coordinates ☆

Renjie Chen [a,*], Craig Gotsman [b]

[a] Max Planck Institute for Informatics, Saarbrucken, Germany
[b] Jacobs Technion–Cornell Institute, Cornell Tech, New York, USA

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ABSTRACT

Harmonic coordinates are widely considered to be perfect barycentric coordinates of a polygonal domain due to their attractive mathematical properties. Alas, they have no closed form in general, so must be numerically approximated by solving a large linear equation on a discretization of the domain. The alternatives are a number of other simpler schemes which have closed forms, many designed as a (computationally) cheap approximation to harmonic coordinates. One test of the quality of the approximation is whether the coordinates coincide with the harmonic coordinates for the special case where the polygon is close to a circle (where the harmonic coordinates have a closed form – the celebrated Poisson kernel). Coordinates which pass this test are called "pseudo-harmonic". Another test is how small the differences between the coordinates and the harmonic coordinates are for "real-world" polygons using some natural distance measures.

We provide a qualitative and quantitative comparison of a number of popular barycentric coordinate methods. In particular, we study how good an approximation they are to harmonic coordinates. We pay special attention to the Moving-Least-Squares coordinates, provide a closed form for them and their transfinite counterpart (i.e. when the polygon converges to a smooth continuous curve), prove that they are pseudo-harmonic and demonstrate experimentally that they provide a superior approximation to harmonic coordinates.

1. Introduction

1.1. Polygon barycentric coordinates

Barycentric coordinates were developed primarily for interpolation purposes, the most common scenario being the interpolation of a real function given on the boundary of a two-dimensional polygon, where the values of the function are specified on the polygon vertices, and assumed to vary linearly between these values along the edges. The objective is then to associate with any interior point in the polygon a real value which is some natural combination of the values given at the vertices.

More precisely, let P be a planar polygon with vertices $p_j = (x_j, y_j)$, $j = 1, \ldots, n$. Given real values f_j at p_j, what should be the value $f(x, y)$ associated with a point $(x, y) \in \text{int}(P)$ (interior to P)? To answer this, we associate with each vertex

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* Corresponding author.
E-mail address: renjie.c@gmail.com (R. Chen).

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p_j a *barycentric coordinate function* $B_j(x, y)$ which satisfies a number of natural conditions, and then define

$$f(x, y) = \sum_{j=1}^{n} f_j B_j(x, y). \tag{1}$$

The conditions that the B_j are required to satisfy are:

C1. Non-negativity: $B_j(x, y) \geq 0,\ j = 1, \ldots, n,\ \forall(x, y) \in \text{int}(P)$
C2. Constant precision: $\sum_{j=1}^{n} B_j(x, y) = 1,\ \forall(x, y) \in \text{int}(P)$
C3. Linear precision: $\sum_{j=1}^{n} x_j B_j(x, y) = x,\ \sum_{j=1}^{n} y_j B_j(x, y) = y,\ \forall(x, y) \in \text{int}(P)$
C4. Lagrange property: $B_j(p_k) = \delta_{jk}$

The main advantage of using barycentric coordinates is that the coordinate functions $B_j(x, y)$ depend only on the polygon P, and not on the f_j, so may be pre-computed. Thus a change in any of the f_j can be reflected easily in $f(x, y)$ as a simple linear combination.

In the field of computer graphics, barycentric coordinates have been used to generate mappings between two-dimensional regions by associating a 2D vector value $q_j = (u_j, v_j)$ with each vertex of P instead of the usual scalar value f_j. This means that the edges of the *source* polygon $P = (p_1, \ldots, p_n)$ are linearly mapped to the edges of the *target* polygon $Q = (q_1, \ldots, q_n)$ and the barycentric coordinate functions define the image of an interior point $(x, y) \in \text{int}(P)$:

$$u(x, y) = \sum_{j=1}^{n} u_j B_j(x, y)$$
$$v(x, y) = \sum_{j=1}^{n} v_j B_j(x, y)$$

The barycentric mapping inherits the properties of the coordinate functions used. Over the years, many recipes for $B_j(x, y)$ have been proposed, the simplest and most well-known being the Laplace (also called *discrete harmonic* or *cotangent*) (Pinkall and Polthier, 1993), mean value (Floater, 2003; Hormann and Floater, 2006) and Wachspress (1975) coordinates. These have simple closed-form expressions for any interior point, so are easy to compute. Alas, they are so simple that they do not behave well on domains with complicated shapes, most notably the Laplace and Wachspress coordinates generate very bad mappings of non-convex domains. Indeed, for many polygons (even convex) they actually violate condition C1 (non-negativity).

Probably the most desirable barycentric coordinate functions are those which generate the unique harmonic mapping between the source and target with the given piecewise linear boundary conditions. Harmonic mappings are desirable since they have many attractive mathematical properties beyond C1–C4, such as smoothness, satisfaction of maximum and minimum principles, the mean-value property and minimization of Dirichlet energy. Another very important property is that harmonic mappings onto convex polygons are guaranteed to be bijective. When used in the Finite-Element Method (FEM), harmonic elements are considered a natural generalization of the linear basis functions on triangles and the bilinear basis functions on quads, and retain almost all their desirable properties even for non-convex elements (Martin et al., 2008). For all these reasons, harmonic maps of polygonal domains have been used in many 2D and 3D deformation applications (Ben-Chen et al., 2009; Schneider and Hormann, 2015). Unfortunately, the harmonic barycentric coordinate functions (first used in Joshi et al., 2007) have no closed form for general polygons and must be computed numerically by solving a discrete Laplace equation with appropriate Dirichlet boundary conditions on P. This requires a finite-element (FEM) discretization of the interior of P, typically by dense triangulation. Although sparse, the resulting linear system, whose size is proportional to the number of finite elements, can be very large, so slow to solve. The result is a piecewise-linear approximation to the coordinate functions on P, albeit with perfect satisfaction of the boundary conditions. Alternative numerical methods (e.g. Weber and Gotsman, 2010) use boundary elements (BEM), which results in perfect harmonic functions on the domain at the price of approximate satisfaction of the boundary conditions. These methods also involve the solution of a dense (though not too large) system of linear equations. Beyond the significant computation complexity, another disadvantage of the harmonic coordinates relative to the simple others mentioned above is that it is impossible to compute the coordinates of just a *single* point in the domain without computing the coordinates of *all* (sampled) domain points.

Since the harmonic barycentric coordinates are so important, a number of methods whose primary objective is to approximate the harmonic coordinates, were invented. These include the mean-value coordinates and others, details of which we will provide later. Of special interest are the so-called Moving Least-Squares (MLS) coordinates (Manson and Schaefer, 2010), which have a closed form requiring not much more computation than, say, mean-value coordinates, but as we will show, generate a superior approximation to harmonic coordinates. The interested reader is also referred to the recent survey by Floater (2015) for a comprehensive overview of many barycentric coordinate recipes.

A simple test for the quality of approximation of a given barycentric coordinate scheme to the harmonic coordinates is the behavior of the coordinates in the special case where the polygon has a very large number of vertices and converges