Computer Aided Geometric Design 44 (2016) 36–55

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Smooth bivariate shape-preserving cubic spline approximation ☆

Victoria Baramidze

Western Illinois University, 1 University Circle, Macomb, IL 61455, USA

ARTICLE INFO

Article history:
Received 19 October 2015
Received in revised form 4 April 2016
Accepted 25 April 2016
Available online 29 April 2016

Keywords:
Shape preservation
Cubic splines
Bivariate approximation

ABSTRACT

Given a piece-wise linear function defined on a type I uniform triangulation we construct a new partition and define a smooth cubic spline that approximates the linear surface and preserves its shape. The key piece is a new macro-element that has the ability to combine six independent gradients coming together at an interior vertex in a smooth yet shape-preserving fashion. The shape of the resulting spline surface follows local changes in the shape of the piece-wise linear interpolant without overshooting. We prove that convexity, positivity and monotonicity of the spline depend on the local data only. Computational scheme for Bernstein–Bezier spline coefficients is local and fast. Numerical examples highlight unique shape-preserving properties of the spline.

1. Introduction

The concept of shape-preservation rises naturally in data fitting problems. Most often we wish that approximating curves and surfaces preserve positivity, monotonicity and convexity of the data. Various methods have been proposed for constructing shape-preserving spline surfaces. For example, the problem of interpolating scattered positive data is solved by positive splines minimizing a thin-plate energy functional in Utreras (1985). Iterative algorithms are proposed to exploit a variational approach with positivity constraints in Kouibia and Pasadas (2003) and Lai and Meile (2015). Local gradient adjusting methods for non-negative interpolation of scattered data in C^1 macro-element spaces, Powell–Sabin and Clough–Tocher splits, are presented in Schumaker and Speleers (2010). In Carlson and Fritsch (1985) authors develop an algorithm for monotone C^1 piecewise bicubic interpolation on a rectangular mesh. This work is continued in Carlson and Fritsch (1989) by presenting a simplified algorithm producing visually pleasing monotone interpolant. Box splines are studied in Chui et al. (1989), where the estimates for grid-size are obtained to guarantee convexity, monotonicity and positivity of solutions. A degree adaptive method for shape-preserving interpolation over a rectangular grid is presented in Costantini and Fontanella (1990). In Costantini and Manni (1991) the method for construction of differentiable interpolating surfaces over rectangular grids produces co-monotone results. In Schmidt and Hess (1993) interpolation of data sets given on rectangular grids is performed by rational bicubic C^2 splines preserving S-convex, monotone, or positive data. Cubic splines on quadrangulated rectangular grids have been successfully used to define monotone surfaces by requiring linearity of certain cross-boundary derivatives in Schumaker and Han (1997). An extension of Clough–Tocher macro-element allows for a construction of interpolating polynomial splines with surface tension controlled by adaptive polynomial degrees, (Costantini and Manni, 1999). An algorithm for convexity-preserving interpolation of scattered data based on choosing nodal gradients in feasible regions

☆ This paper has been recommended for acceptance by Larry Schumaker.
E-mail address: V-Baramidze@wiu.edu.

http://dx.doi.org/10.1016/j.cagd.2016.04.006
0167-8396/

is presented in Leung and Renka (1999). Cubic L_1 smoothing spline on tensor-product grids in Gilsinn and Lavery (2002) demonstrate promising shape-preserving behavior. An energy functional alternative to the one used in Gilsinn and Lavery (2002) is tested in Witzgall et al. (2006) over Clough–Tocher splits of general irregular triangulations. The ideas of ℓ_1 minimization in Lavery (2001) are adapted for general scattered data triangulations and, compared to thin-plate minimal energy and penalized least squares solutions, experiments demonstrate superior shape-preserving properties of the L_1 splines in Lai and Wenston (2004). Iterative knot insertion algorithm generating a sequence of shape-preserving approximants is given in Kuijt and Damme (2001). In Schumaker and Speleers (2011) a search for a convex spline solution is formulated as a quadratic programming problem where convexity is enforced by including appropriate side conditions on the coefficients of the spline. Rational bi-quadratic splines preserving the shape of 3D positive and convex data are used in Hussain et al. (2011).

Often, researches make global assumptions about the shape of given data, i.e. monotonicity, positivity or convexity, and design algorithms for constructing surfaces preserving the particular feature globally. Many methods are based on a variational approach (Utreras, 1985; Kouibia and Pasadas, 2003; Lai and Meile, 2015; Gilsinn and Lavery, 2002; Lai and Wenston, 2004; Witzgall et al., 2006; Schumaker and Speleers, 2011) resulting either in a large system of equations or an iterative algorithm.

Constructions based on local information and leading to computationally attractive local schemes have been successfully employed as well, see for example (Schumaker and Speleers, 2010; Costantini and Manni, 1991, 1999; Manni, 2001). Macro-element spaces have been extensively used in development of shape-preservation methods, see for example (Willemans and Dierckx, 1994, 1995; Schmidt, 1999; Li, 1999; Lai, 2000; Carnicer et al., 2009).

In this paper we develop a local approach to shape-preservation. In fact, the goal of the construction is to follow local changes in data, and shape-preservation allows us to do so without overshooting.

Let Δ be a triangulation of the domain Ω with function values given at the vertices of Δ. There are some connected subsets of Δ on which the data are positive, and others, where the data are monotone and/or convex. What we claim and prove is that, if on a subset D of Δ the given vertex data is monotone (convex or positive), then for every $0 < \lambda < 1/6$ there exists a set $D_\lambda \subset D$ on which the constructed spline S_λ is monotone (convex or positive), and $\lim_{\lambda\to 0} D_\lambda = D$.

A C^1 cubic spline is a popular choice for many interpolation/approximation problems. Polynomials of relatively low degree are well understood, and many spline tools, such as some macro-elements, for example, are specifically designed for C^1 cubic splines. Spline theory suggests that some of the C^1 conditions across the edges of a type I uniform triangulation are too restrictive, and one will have a problem controlling a cubic spline constructed over Δ due to these restrictions. A single coefficient may affect the spline over the rest of the triangles. There are various macro-element spaces that remedy this problem: after a refinement each coefficient has a finite number of triangles "around" it to control. What we suggest is not, strictly speaking, a refinement of the given Δ, since it does not preserve its edges, nor does it preserve all of its vertices. It is a refinement of Δ in a sense that the new triangulation $\tilde{\Delta}$ consists of many more triangles than Δ, and the geometry and arrangements of these triangles are intimately connected with the original geometry of Δ. A parameter λ controls the size of triangles, and there is more than one $\tilde{\Delta}$ that works (take any $0 < \lambda < 1/6$). This parameter affects the final look of the spline surface, however the surface is shape-preserving for any value of the parameter in the given range.

The constructed spline satisfies many attractive properties and has a few limitations. First of all, a triangulation Δ is not a triangulation of a scattered data set, it is a type I triangulation. Second, the constructed spline interpolates values of a piece-wise linear function, L, at locations other than vertices, while traditionally we seek splines interpolating data at the vertices of Δ. Finally, as an approximation to a piece-wise linear interpolant, the spline produced by the proposed method has regions of flatness, and this feature may limit practical applications of the construction. To minimize regions of flatness larger values of λ must be used. In fact, preliminary testing demonstrates that the case $\lambda = 1/6$ has similar shape-preserving properties. Since $\tilde{\Delta}$ in this case is significantly different, details of this construction and corresponding proofs will have to be reported else-where. Otherwise (if larger λ's are not satisfactory), it is not too difficult to see that ideas presented here can be used in combination with other spline constructions, extended to parametric surfaces, used on parts of domains, etc.

In Section 2 we present detailed construction of $\tilde{\Delta}$. In Section 3 we describe how the coefficients of the spline are computed. We study linearity, positivity, monotonicity, convexity of the spline in Section 4, using results in Lai and Schumaker (2007) to connect the behavior of a BB-polynomial to its coefficients. In Section 5, we discuss numerical experiments performed with shape-preserving splines, and follow up with conclusions in Section 6.

2. Repartitioning a type I uniform triangulation

Let Δ be a type I uniform triangulation of a planar convex domain Ω with the vertices forming a set V, and let $\tilde{\Delta}$ denote the new triangulation. Divide triangles of Δ into two groups, as marked by 1 and 2 in Fig. 1, left (there are two choices for grouping, the key is to alternate triangles from different groups). Fix $0 < \lambda < 1/6$ and associate a weight $w = \lambda$ with triangles marked by 1, and a weight $w = 2\lambda$ with triangles marked by 2. In every triangle $\tau^{(i)} \in \Delta$ with vertices v_1, v_2, v_3 define three points

$$w_1^{(i)} = (1 - 2w)v_1 + wv_2 + wv_3,$$
$$w_2^{(i)} = wv_1 + (1 - 2w)v_2 + wv_3,$$