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Subdividing barycentric coordinates

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ABSTRACT

Barycentric coordinates are commonly used to represent a point inside a polygon as an affine combination of the polygon's vertices and to interpolate data given at these vertices. While unique for triangles, various generalizations to arbitrary simple polygons exist, each satisfying a different set of properties. Some of these generalized barycentric coordinates do not have a closed form and can only be approximated by piecewise linear functions. In this paper we show that subdivision can be used to refine these piecewise linear functions without losing the key barycentric properties. For a wide range of subdivision schemes, this generates a sequence of piecewise linear coordinates which converges to non-negative and C^1 continuous coordinates in the limit. The power of the described approach comes from the possibility of evaluating the C^1 limit coordinates and their derivatives directly. We support our theoretical results with several examples, where we use Loop or Catmull–Clark subdivision to generate C^1 coordinates, which inherit the favourable shape properties of harmonic coordinates or the small support of local barycentric coordinates.

1. Introduction

Suppose we are given a planar n-sided simple polygon $\Omega \subset \mathbb{R}^2$ with $n \geq 3$ vertices $v_1, \ldots, v_n \in \mathbb{R}^2$. For any $p \in \mathbb{R}^2$, the values

$$\big[b_1(p), \ldots, b_n(p)\big] = b(p) \in \mathbb{R}^n$$

are called *barycentric coordinates* of p with respect to Ω, if

$$\sum_{i=1}^{n} b_i(p) = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} b_i(p) v_i = p. \tag{1}$$

Non-negativity is sometimes mentioned as an additional condition (Floater et al., 2006), but since this precludes the existence of barycentric coordinates at points outside the convex hull of the vertices v_i, we prefer to consider the conditions in (1) as the *defining* properties and regard non-negativity as a *desirable* property only.

It is well known (Möbius, 1827) that the barycentric coordinates of p are unique for $n = 3$, when Ω is a triangle, and they are non-negative if and only if $p \in \Omega$ in this case. Instead, for $n > 3$ the conditions in (1) describe an $(n-3)$-dimensional

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affine subspace of $\mathbb{R}^n$ from which $b(p)$ can be chosen. For example, Waldron (2011) suggests to consider barycentric coordinates with minimal ℓ_2-norm and derives an explicit formula for computing them. He further shows that these *affine barycentric coordinates* are non-negative in a convex region that contains the barycentre $\bar{v} = (v_1 + \cdots + v_n)/n$ of Ω. Another example are Floater's *shape preserving coordinates* (Floater, 1997) which are well-defined and non-negative for any p in the kernel of Ω and have been used successfully for mesh parameterization (Floater, 1997) and morphing (Floater and Gotsman, 1999).

Both applications rely on *pointwise* barycentric coordinates, in the sense that $b(p)$ with the properties in (1) must be determined for a single point p in the kernel of some polygon Ω. Instead, other applications, like geometric modelling (Loop and DeRose, 1989), colour interpolation (Meyer et al., 2002), rendering (Hormann and Tarini, 2004), shape deformation (Ju et al., 2005), and image warping (Warren et al., 2007), require barycentric coordinates for all $p \in \Omega$ and consider $b(p)$ as a function of p over Ω. In this setting, the individual *barycentric coordinate functions* $b_i : \Omega \to \mathbb{R}$ must satisfy the *Lagrange property*

$$b_i(v_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 1, \ldots, n \tag{2}$$

in addition to the defining conditions in (1), so that the function $f : \Omega \to \mathbb{R}^d$ with

$$f(p) = \sum_{i=1}^{n} b_i(p) f_i \tag{3}$$

interpolates the data $f_1, \ldots, f_n \in \mathbb{R}^d$ at the vertices $v_1, \ldots, v_n$. Most applications further expect the barycentric coordinate functions to be *smooth*, so that the *barycentric interpolant* f in (3) is C^1 or even C^2 continuous. And for some applications it is crucial that the coordinates are *non-negative*, because this guarantees that the interpolated values $f(p)$ are contained in the convex hull of the data.

1.1. Related work

Wachspress (1975) was the first to describe a construction of rational barycentric coordinate functions for *convex* polygons in the context of generalized finite element methods, but these *Wachspress coordinates* are not well-defined for arbitrary simple polygons. The same holds for *discrete harmonic coordinates*, which arise from the classical piecewise linear finite element approximation to Laplace's equation (Strang and Fix, 2008) and have been applied for computing discrete minimal surfaces (Pinkall and Polthier, 1993) and mesh parameterization (Eck et al., 1995). *Mean value coordinates* (Floater, 2003) overcome this drawback, as they are well-defined even for sets of nested simple polygons and for any $p \in \mathbb{R}^2$ (Hormann and Floater, 2006). However, mean value coordinates can be negative inside concave polygons, and the same is true for *metric* (Sukumar and Malsch, 2006), *moving least squares* (Manson and Schaefer, 2010), *Poisson* (Li and Hu, 2013), and *cubic mean value coordinates* (Li et al., 2013). Positivity inside arbitrary simple polygons is guaranteed by *positive mean value* (Lipman et al., 2007) and *positive Gordon–Wixom coordinates* (Manson et al., 2011), but both constructions deliver only C^0 continuous coordinate functions.

All the aforementioned constructions provide *closed-form coordinates*, which can be evaluated exactly for any $p \in \Omega$ in a finite number of steps. At the same time, neither of these coordinates are smooth *and* positive inside non-convex polygons. So far the only barycentric coordinates known to have both properties are the *harmonic* (Joshi et al., 2007), *maximum entropy* (Hormann and Sukumar, 2008), and *local barycentric coordinates* (Zhang et al., 2014), but they all are *computational coordinates* in the sense that they lack a closed-form expression and must be treated numerically.

For example, harmonic coordinates can be approximated by using the *complex variable boundary element method* (Weber and Gotsman, 2010, Sec. 6.1) or the *method of fundamental solutions* (Martin et al., 2008, Sec. 5). The advantage of both approaches is that the resulting coordinates are smooth and harmonic and can be written in closed form after initially solving a rather small but dense linear system, but they only approximate the piecewise linear boundary conditions and thus do not satisfy the Lagrange property.

Another common strategy for computing harmonic coordinates (Eck et al., 1995; Joshi et al., 2007) is first to create a dense triangulation of Ω, then to fix the barycentric coordinates of the boundary vertices according to the Lagrange property (2) and such that the coordinates are linear along the edges of Ω, and finally to determine the coordinates at the interior vertices using the standard finite element discretization of the Laplace equation with Dirichlet boundary conditions. This approach is quite efficient, because it only requires solving a sparse linear system, but the resulting coordinate functions are merely *piecewise linear* approximations of the true harmonic coordinates. Local barycentric coordinates are approximated similarly, except that computing the coordinates at the interior vertices is more involved as it leads to a convex optimization problem with a non-smooth target function (Zhang et al., 2014, Sec. 4). However, the advantage of the resulting coordinate functions is that their support is smaller than the support of harmonic coordinate functions.

In both cases a *global* problem is solved to determine the barycentric coordinates for all interior vertices simultaneously. In contrast, maximum entropy coordinates are computed for any $p \in \Omega$ by solving a *local* convex optimization problem, which in turn can be done very efficiently with Newton's method (Hormann and Sukumar, 2008, Sec. 5).